

Cayley submanifolds of Calabi-Yau 4-folds.

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Dedicated to Jim Eells

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Abstract: Our main results are: (1) The complex and Lagrangian points of a non-complex and non-Lagrangian $2n$ -dimensional submanifold $F : M \rightarrow N$, immersed with parallel mean curvature and with equal Kähler angles into a Kähler-Einstein manifold (N, J, g) of complex dimension $2n$, are zeros of finite order of $\sin^2 \theta$ and $\cos^2 \theta$ respectively, where θ is the common J -Kähler angle. (2) If M is a Cayley submanifold of a Calabi-Yau (CY) manifold N of complex dimension 4, then $\bigwedge_+^2 NM$ is naturally isomorphic to $\bigwedge_+^2 TM$. (3) If N is Ricci-flat (not necessarily CY) and M is a Cayley submanifold, then $p_1(\bigwedge_+^2 NM) = p_1(\bigwedge_+^2 TM)$ still holds, but $p_1(\bigwedge_-^2 NM) - p_1(\bigwedge_-^2 TM)$ may describe a residue on the J -complex points, in the sense of Harvey and Lawson. We describe this residue by a PDE on a natural morphism $\Phi : TM \rightarrow NM$, $\Phi(X) = (JX)^\perp$, with singularities at the complex points. We give an explicit formula of this residue in a particular case. When (N, I, J, K, g) is a hyper-Kähler manifold and M is an I -complex closed 4-submanifold, the first Weyl curvature invariant of M may be described as a residue on the J -Kähler angle at the J -Lagrangian points by a Lelong-Poincaré type formula. We study the almost complex structure J_ω on M induced by F .

1 Introduction

The role of the complex and anti-complex points on the topology-geometry of closed non-complex minimal surfaces immersed into complex Kähler surfaces has been studied in [28], [9], [8], and [31]. In these papers, it is proved that the set $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$ of complex and anti-complex points is a set of isolated points, and each of such points is of finite order. The order of the complex and anti-complex points is defined as a multiplicity of a zero of $(1 \pm \cos \theta)$, where θ is the Kähler angle, and adjunction formulas were obtained in [28], [8] and [31]:

$$-\sum_{p \in \mathcal{C}^-} \text{order}(p) - \sum_{p \in \mathcal{C}^+} \text{order}(p) = \mathcal{X}(M) + \mathcal{X}(NM) \quad (1.1)$$

$$\sum_{p \in \mathcal{C}^-} \text{order}(p) - \sum_{p \in \mathcal{C}^+} \text{order}(p) = F^* c_I(N)[M]. \quad (1.2)$$

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The proofs of these formulas come, respectively, from the following PDEs of second order on the cosine of the Kähler angle, with singularities at complex and anti-complex points:

$$\frac{1}{2}\Delta \log \sin^2 \theta = (K^M + K^\perp) \quad (1.3)$$

$$\frac{1}{2}\Delta \log \left(\frac{1 + \cos \theta}{1 - \cos \theta} \right) = -Ricci^N(e_1, e_2), \quad (1.4)$$

where K^M and K^\perp are respectively the Gaussian curvature of M and the curvature of the normal bundle NM , and e_1, e_2 is a direct orthonormal frame of M . Therefore, (1.1) and (1.2) are formulas that describe some polynomials of topological invariants of the immersed surface, normal bundle and ambient space, as residue formulas of certain functions that have singularities at those special points.

In higher dimensions, the papers [17], [29], [30] show how Pontrjagin classes and Euler classes of a closed (generic) submanifold M of a complex manifold (N, J) are carried by subsets of CR-singular points, that is, points with sufficiently many complex directions. The investigation of complex tangents on a m -dimensional submanifold M embedded into a Kähler manifold N of complex dimension m is very much justified, by the well known embedding theorem of Whitney. More generally, if N has a calibration Ω of rank m (see definitions in [12]) and M is not Ω -calibrated we may expect that Ω -calibrated points may have a similar role ([21]). Minimality of M should guarantee the order of such points to be finite.

In [13], [14] a general framework is shown to obtain this sort of geometric residues, inspired by the above examples. Given two Riemannian vector bundles (E, g_E) , (F, g_F) over M , of the same rank m , with Riemannian connections ∇^E and ∇^F , and a bundle map $\Phi : F \rightarrow E$, degenerated at a set of points Σ , we may compare a m -characteristic classe Ch of E and the one of F , describing these invariants using the curvature tensors with respect to ∇^F and ∇^E , via Chern-Weil theory. Φ induces on F a singular connection $\nabla' = \Phi^{-1*}\nabla^E$, Riemannian for a degenerated metric, and that makes Φ a parallel isometric bundle map, but R' and $Ch(R')$ can be smoothly extended to Σ by the identities $R'(X, Y, Z, W) = g_E(R^E(X, Y)\Phi(Z), \Phi(W))$, $Ch(R') = Ch(R^E)$. The difference $Ch(R') - Ch(R^F)$ is of the form dT where T is a transgression form with singularities along Σ . If Σ is sufficiently small and regular, the Stokes theorem reads $\int_{M \sim V_\epsilon(\Sigma)} dT = - \int_{\partial V_\epsilon(\Sigma)} T$, where $V_\epsilon(\Sigma)$ is a tubular neighbourhood of Σ of radius ϵ , and letting $\epsilon \rightarrow 0$ may describe $Ch(E) - Ch(F)$ as a residue of T along Σ and expressed in terms of the zeros of Φ .

Inspired in this framework, the present paper shows some formulas of the type (1.3)-(1.4) for 4-dimensional submanifolds of certain Kähler manifolds. As we will see, to workout such formulas in dimension > 2 is considerably more difficult then in the surface case.

We study the set \mathcal{C} of complex points and the set \mathcal{L} of Lagrangian points of non-holomorphic and non-Lagrangian immersed submanifolds $F : M \rightarrow N$ of real dimension $2n$ of a Kähler-Einstein (KE) manifold of complex dimension $2n$, namely if F is immersed with equal Kähler angles (e.k.a.s). A natural bundle map $\Phi : TM \rightarrow NM$, $\Phi(X) = (JX)^\perp$, is defined and was first used by Webster. Φ is degenerated at points with complex directions, and has maximum norm at Lagrangian points, where it is an isometry. If F has e.k.a.s, Φ is conformal with

$\|\Phi(X)\|^2 = \sin^2 \theta \|X\|^2$ where θ is the common Kähler angle, and away from \mathcal{L} , one can define smooth almost complex structures J_ω on M and J^\perp on the normal bundle NM that are naturally inherited from the ambient space, and they coincide with the induced complex structure at complex points. These almost complex structures, with the Kähler angle, will be fundamental for our formulas. In section 3, if $n = 2$ we study J_ω .

If $n = 2$ and (N, J, g) is Ricci-flat KE, M is a Cayley submanifold if it is minimal and with equal Kähler angles. If N is Calabi-Yau these Cayley submanifolds are calibrated by one of the S^1 -family of Cayley calibrations ([12],[16]). The Cayley calibrations Ω do not specify the complex or the Lagrangian points, but induce a natural isomorphism $\Omega^M : \bigwedge_+^2 TM \rightarrow \bigwedge_+^2 NM$, $\langle \Omega^M(X \wedge Y), U \wedge V \rangle = \Omega(X, Y, U, V)$ (see Prop.3.2)

In Section 4 we prove that complex and Lagrangian points of a n -submanifold with parallel mean curvature are zeros of a system of complex-valued functions that satisfy a second-order partial differential system of inequalities of the Aronszajn type, and so, if the submanifold is not complex or Lagrangian, they are zeros of finite order. These inequalities are obtained from some estimates on the Laplacian of the pull-back of the Kähler form of N by F , and on the Laplacian of $\Phi : TM \rightarrow NM$. Furthermore, the sets \mathcal{C} and \mathcal{L} have Hausdorff codimension at least 1, and if M is closed and $n = 2$, \mathcal{L} is a set of Hausdorff codimension at least 2.

In Section 5 we prove the following residue-type formula, in the same spirit as formulas (1.3) and (1.4):

Theorem 1.1. *If $F : M \rightarrow N$ is a non- J -holomorphic Cayley submanifold immersed into a 4-fold Ricci-flat Kähler manifold (not necessarily Calabi-Yau), the following equalities hold, for some representatives in the cohomology classes of M :*

$$p_1(\bigwedge_+^2 NM) = p_1(\bigwedge_+^2 TM) \quad (1.5)$$

$$p_1(\bigwedge_-^2 NM) = p_1(\bigwedge_-^2 TM) + \frac{1}{\pi^2} d\eta \quad (1.6)$$

where $\eta = \eta(\Phi)$ is a 3-form, defined away from the complex points, which is given by

$$\eta(\Phi) = -\frac{1}{4} \left\langle \Phi^{-1} \nabla \Phi \wedge (\Phi(R^\perp) + R^M + \frac{1}{3} [\Phi^{-1} \nabla \Phi, \Phi^{-1} \nabla \Phi]) \right\rangle \quad (1.7)$$

where $\Phi(R^\perp) : \bigwedge^2 TM \rightarrow \bigwedge^2 TM$ is given by $\Phi(R^\perp)(X, Y)(Z) = \Phi^{-1} R^\perp(X, Y)\Phi(Z)$. Furthermore:

- (A) If F has no complex points TM and NM have the same Pontrjagin and Euler classes.
- (B) If $d\Phi = 0$, or if $g(\nabla_X \Phi(Y), \Phi(Z))$ is skew symmetric on (Y, Z) , then θ is constant and $\Phi : TM \rightarrow NM$ is a parallel homothetic diffeomorphism.
- (C) If $g(\nabla_X \Phi(Y), \Phi(Z))$ is symmetric on (Y, Z) , or if $\bar{R}(X, Y)\Phi = 0$, where \bar{R} is the curvature tensor of $TM^* \otimes NM$, then $p_1(\bigwedge_-^2 TM) = p_1(\bigwedge_-^2 NM)$ holds, or equivalently M and NM have the same Pontrjagin and Euler classes.

We will say that F has *regular homogeneous complex points*, if $\mathcal{C} = \bigcup_i \Sigma_i$ is a disjoint finite union of closed submanifolds Σ_i of dimension $d_i \leq 3$, and for each i , on a neighbourhood V of Σ_i in M , $\sin \theta = f_i^{r_i}$ where r_i is the common order of the zeros of Φ (and of $\|\Phi\|$) along Σ_i ,

and f_i is a nonnegative continuous function, smooth on $V \sim \mathcal{C}$, such that $\|\nabla f_i\|$ exists as a positive C^μ function on all V , with $\mu \geq r_i + 2$ and the flow of $X_{f_i} = \frac{\nabla f_i}{\|\nabla f_i\|^2}$ can be extended to $G_{t_0} = \{(p, w) \in N\Sigma_i : \|w\| < t_0\}$ as a $C^{\mu+1}$ diffeomorphism $\xi : G_{t_0} \rightarrow V$. That is, X_{f_i} is a multivalued vector field at points $p \in \Sigma_i$, with sublimits spanning all $T_p\Sigma_i^\perp$ and for each u unit vector of $T_p\Sigma_i^\perp$ it is defined an integral curve $\gamma_{(p,u)}(t) = \xi(p, tu)$ with $\gamma_{(p,u)}(0) = p$ and $\gamma'_{(p,u)}(0) = \frac{u}{c(p)}$, where $c(p) = \|\nabla f_i\|(p)$. This flow map ξ defines for each sufficiently small $\epsilon > 0$ a diffeomorphism from $C_\epsilon = \{(p, w) \in N\Sigma_i : \|w\| = \epsilon\}$ onto $f_i^{-1}(\epsilon)$. Furthermore, for each sufficiently small coordinate chart y of Σ we have a Fermi-type coordinate chart x of V of class $C^{\mu+1}$, extending y and satisfying $f_i = \sqrt{x_{d_i+1}^2 + \dots + x_4^2}$ (see Prop. 5.8). Examples of such functions f_i are the distance function σ to a submanifold Σ_i . Let $\pi : N^1\Sigma_i := C_1 \rightarrow \Sigma_i$, $\pi(p, u) = p$, and $S(p, 1)$ the unit sphere of $T_p\Sigma_i^\perp \subset T_pM$. For $u \in S(p, 1)$ and $X \in T_pM$, set $X^{\perp u} = X - g(X, u)u$, and define $\varsigma(u)(X) = (\nabla_u \tilde{X})^\perp \in T_pM$ where \tilde{X} is any smooth section of TM with $\tilde{X}_p = X$.

Corollary 1.1. *Assume M is compact and F in Theorem 1.1 has regular homogeneous complex points of order r_i on Σ_i . Let $\tilde{\Phi} = \frac{\Phi}{\|\Phi\|}$ and set for each $(p, u) \in N^1\Sigma_i$ and X vector field on M ,*

$$\begin{aligned} \Upsilon_i(p, u) &:= \nabla_{u^{r_i}}^{r_i} \Phi(p), & \Psi_i(p, u)(X_p) &:= \nabla_{u^{r_i-1}, X}^{r_i} \Phi(p), \\ G_i(p, u)(X_p) &:= \nabla_{u^{r_i}, X}^{(r_i+1)} \Phi(p) + r_i \Psi_i(p, u)(\varsigma(u)(X_p)) \end{aligned} \quad (1.8)$$

defining smooth sections Υ_i of $\pi^{-1}(TM^* \otimes NM)$ and Ψ_i, G_i of $\pi^{-1}(TM^* \otimes (TM^* \otimes NM))$. Then there exist the following limits

$$\lim_{\epsilon \rightarrow 0} \tilde{\Phi}(\xi(p, \epsilon u)) = \frac{1}{r_i! c(p)^{r_i}} \Upsilon_i(p, u), \quad (1.9)$$

$$\lim_{\epsilon \rightarrow 0} \epsilon \nabla_X \tilde{\Phi}(\xi(p, \epsilon u)) = \frac{1}{(r_i - 1)! c(p)^{r_i-1}} \Psi_i(p, u)(X^{\perp u}) \quad (1.10)$$

$$\text{if } X \perp \nabla f_i \text{ near } p, \quad \lim_{\epsilon \rightarrow 0} \nabla_X \tilde{\Phi}(\xi(p, \epsilon u)) = \frac{1}{r_i! c(p)^{r_i}} G_i(p, u)(X_p), \quad (1.11)$$

with $\frac{1}{r_i! c(p)^{r_i}} \Upsilon_i(p, u) : T_pM \rightarrow NM_p$ an isometry. Furthermore, set

$$T_i^{(1)}(p, u)(X) = \Upsilon_i(p, u)^{-1} \circ \Psi_i(p, u)(X) \quad T_i^{(0)}(p, u)(X) = \Upsilon_i(p, u)^{-1} \circ G_i(p, u)(X).$$

Then

$$\begin{aligned} & p_1(\wedge_-^2 NM)[M] - p_1(\wedge_-^2 TM)[M] = \\ &= \sum_{i: d_i=2} -\frac{r_i}{4} \int_{\Sigma_i} \left(\int_{S(p,1)} c(p)^{-1} \langle T_i^{(1)}(p, u) \wedge (\Upsilon(p, u)(R^\perp) + R^M) \rangle (*u) d_{S(p,1)}(u) \right) d_{\Sigma_i}(p) \\ &+ \sum_{k=0}^3 \sum_{i: d_i=k} \sum_{\alpha+\beta+\gamma=3-k} -\frac{r_i^{3-k}}{12} \int_{\Sigma_i} \left(\int_{S(p,1)} c(p)^{-1} \langle T_i^{(\alpha)}(p, u) \wedge [T_i^{(\beta)}(p, u), T_i^{(\gamma)}(p, u)] \rangle (*u) d_{S(p,1)}(u) \right) d_{\Sigma_i}(p). \end{aligned}$$

In section 6 we prove that if M is a J -complex submanifold and N is Ricci-flat, then (1.5) still holds. Moreover, if $c_1(M) = 0$, then $\wedge_+^2 TM$ and $\wedge_+^2 NM$ are both flat, and $\wedge_-^2 TM$ and

$\bigwedge_-^2 NM$ are both anti-self-dual.

If $N = (N, I, J, K, g)$ is hyper-Kähler (HK) of complex dimension 4, and M is an I -complex submanifold of complex dimension 2, then, considering on N the complex structure J , M is a Cayley submanifold with a J -Kähler angle θ that can assume any value. Furthermore, M has a hyper-Hermitian structure $(M, I, J_{\omega_J}, J_{\omega_K}, g)$, defined away from totally complex points. More generally, if M is an " I -Kähler" Cayley submanifold of a Ricci-flat Kähler 4-fold (N, J, g) , i.e., locally on an open dense set of $M \sim \mathcal{L}$, a smooth Kähler structure I exists and that anti-commutes with J_{ω} , then we conclude (in subsection 3.2) that the J -Kähler angle θ also satisfies the PDE

$$\Delta \log \cos^2 \theta = s^M \quad (1.12)$$

where s^M is the scalar curvature of M . If M is closed, this is a residue-type formula for the first curvature invariant of Weyl of M , $\kappa_2(M) = \frac{1}{2} \int_M s^M \text{Vol}_M$, in terms of the zero set Σ of $\cos \theta$, which is the set of the J -Lagrangian points \mathcal{L} of M . We prove in section 7:

Proposition 1.1. *If N is HK and M is a non-totally complex closed I -Kähler submanifold with I -Kähler form ω_I , then there exist a locally finite union of irreducible analytic subvarieties of complex codimension 1 (i.e analytic surfaces) Σ_i and integers a_i such that $\Sigma = \bigcup_i \Sigma_i$ where $\cos \theta$ vanish at homogeneous order a_i along Σ_i and a formula of Lelong-Poincaré type in terms of characteristic divisors exist: $\frac{1}{\pi} \kappa_2(M) = - \sum_i a_i \int_{\Sigma_i} \omega_I$.*

If I does not exist globally on M , we still can obtain a residue formula under some conditions, and a removable high rank singularity theorem (see Proposition 7.1 and Corollary 7.1).

In section 8 we give some examples of complete non-linear Cayley submanifolds of (\mathbb{R}^8, J_0, g_0) , with no complex J_0 -points, with only one complex point, with a 2-plane set of complex points, or with a 2-plane set of Lagrangian points. They are all holomorphic for some other complex structure of \mathbb{R}^8 . We also observe that all submanifolds, and in particular coassociative ones, that are graphs of maps $f : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, do not have J_0 -complex points.

2 The Kähler angles

We recall the notion of Kähler angles introduced in [22], [23] and [26] for an immersed $2m$ -submanifold $F : M \rightarrow N$ of a Kähler manifold of complex dimension $2n$ where $m \leq n$. We denote by J and g the complex and Hermitian structure of N and $\omega(X, Y) = g(JX, Y)$ its Kähler form. Let $NM = (dF(TM))^\perp$ be the normal bundle and denote by $(\cdot)^\perp$ the orthogonal projection of $F^{-1}TN$ onto NM . The pullback 2-form $F^*\omega$ defines at each point $p \in M$ the Kähler angles $\theta_1, \dots, \theta_m$ of M , $\theta_\alpha \in [0, \frac{\pi}{2}]$, such that $\cos \theta_1 \geq \dots \geq \cos \theta_m \geq 0$ and $\{\pm i \cos \theta_\alpha\}_{\alpha=1, \dots, m}$ are the eigenvalues of the complex extension $F^*\omega$ to $T_p^c M$. Polar decomposition of the endomorphism $(F^*\omega)^\sharp = |(F^*\omega)^\sharp| J_\omega$ where \sharp is the usual musical isomorphism, defines a partial isometry $J_\omega : TM \rightarrow TM$ with the same kernel \mathcal{K}_ω as $F^*\omega$. Then $M = \bigcup_{k=0}^m \mathcal{L}_{m-k}$ where a point $p \in M$ is in \mathcal{L}_{m-k} iff $\text{Rank}(F^*\omega)_p = 2k$. At each $p \in \mathcal{L}_{m-k}$, we may take an o.n. basis of $T_p M$ of the form $\{X_1, Y_1 = J_\omega X_1, \dots, X_k, Y_k = J_\omega X_k, X_{k+1}, Y_{k+1}, \dots, X_m, Y_m\}$ where $\{X_{k+1}, Y_{k+1}, \dots, X_m, Y_m\}$ is any o.n. basis of \mathcal{K}_ω . If O is an open set of M lying in \mathcal{L}_{m-k} , then X_α, Y_α can be chosen

smoothly on a neighbourhood of each point of O . The complex frame

$$\alpha = Z_\alpha := \frac{X_\alpha - iY_\alpha}{2} \quad \bar{\alpha} = \bar{Z}_\alpha \quad \alpha = 1, \dots, m \quad (2.1)$$

diagonalizes $F^*\omega$, $(F^*\omega)^\sharp(Z_\alpha) = i \cos \theta_\alpha Z_\alpha$, and for $\alpha \leq k$, $Z_\alpha \in T^{(1,0)}M$ w.r.t. J_ω . We will use the greek letters $\alpha, \beta, \mu, \dots$ and their conjugates to denote both the integer in $\{1, 2, \dots, m\}$ it represents or the corresponding complex vector of T^cM above defined in (2.1). If M is orientable then \mathcal{C}^+ and \mathcal{C}^- , are the set of points such that $\cos \theta_\alpha(p) = 1 \ \forall \alpha$, and respectively, J_ω defines the same or the opposite orientation of M . The eigenvalues $\cos \theta_\alpha$ are only locally Lipschitz on M , while the product $2\epsilon(p) \cos \theta_1 \dots \cos \theta_m = \langle F^*\omega^m, Vol_M \rangle$ is smooth everywhere, where $\epsilon(p)$ is the orientation of J_ω for $p \in \mathcal{L}_0$. For E subspace of T_pM set $E^J = E \cap JE$.

Let $\omega^\perp = \omega|_{NM}$ be the restriction of the Kähler form ω of N to the normal bundle NM , and $(\omega^\perp)^\sharp = |(\omega^\perp)^\sharp|J^\perp$ be its polar decomposition. We define the following morphisms

$$\begin{array}{ccc} \Phi : TM & \rightarrow & NM \\ X & \rightarrow & (JX)^\perp \end{array} \quad \begin{array}{ccc} \Xi : NM & \rightarrow & TM \\ U & \rightarrow & (JU)^\top \end{array}$$

Note that $JX = (F^*\omega)^\sharp(X) + \Phi(X)$, and $\Phi(X) = 0$ iff $\{X, JX\}$ is a complex direction of F . Similarly for ω^\perp and Ξ . $\Phi : (TM^J)^\perp \rightarrow NM$, $\Xi : (NM^J)^\perp \rightarrow TM$ are 1-1. Set $2s = \dim(TM^J)$, $2t = \dim(NM^J)$. The o.n. basis $\{U_A, V_A\} = \{U_1, JU_1, \dots, U_t, JU_t, \Phi(\frac{Y_\alpha}{\sin \theta_\alpha}), \Phi(\frac{X_\alpha}{\sin \theta_\alpha})\}$, where α are s.t. $\sin \theta_\alpha \neq 0$, diagonalize ω^\perp , and so $2n = 2m + t - s$, and for $A = \alpha + t - s$, $\sigma_A = \theta_\alpha$ are the non-zero Kähler angles of NM . That is, TM and NM have the same nonzero Kähler angles, and they have the same multiplicity. Only the eigenvalues $\pm i$ of $(F^*\omega)^\sharp$ and of ω^\perp may or not exist and may appear with different multiplicity, t and s , respectively. Set $E_\alpha = \text{span}\{X_\alpha, Y_\alpha\}$, $F_A = \text{span}\{U_A, V_A\}$, and P_{E_α}, P_{F_A} the corresponding orthonormal projections of TM and NM . We use the Hilbert-Schmidt inner products on tensors and forms. We have

$$\begin{aligned} \|F^*\omega\|^2 &= \frac{1}{2} \|(F^*\omega)^\sharp\|^2 = \sum_\alpha \cos^2 \theta_\alpha = \|\omega^\perp\|^2 + (s - t) = \|\omega^\perp\|^2 - 2(n - m) \\ g(\Phi(X), \Phi(Y)) &= (1 - \cos \theta_\alpha \cos \theta_\beta)g(X, Y) \quad \text{for } X \in E_\alpha, Y \in E_\beta \end{aligned} \quad (2.2)$$

$$\begin{aligned} g(\Xi(U), \Xi(V)) &= (1 - \cos \theta_\alpha \cos \theta_\beta)g(U, V) \quad \text{for } U \in F_\alpha, V \in F_\beta \\ \|\Phi\|^2 &= 2 \sum_\alpha \sin^2 \theta_\alpha = \|\Xi\|^2 - 4(n - m) \\ \omega^\perp \circ \Phi &= -\Phi \circ (F^*\omega)^\sharp & (F^*\omega)^\sharp \circ \Xi &= -\Xi \circ \omega^\perp \\ J^\perp \circ \Phi &= -\Phi \circ J_\omega & J_\omega \circ \Xi &= -\Xi \circ J^\perp \quad \text{on } \mathcal{L}_0 \end{aligned} \quad (2.3)$$

$$-\Xi \circ \Phi = \sum_\alpha \sin^2 \theta_\alpha P_{E_\alpha} \quad -\Phi \circ \Xi = \sum_\alpha \sin^2 \theta_\alpha P_{F_\alpha}. \quad (2.4)$$

If $X \in T_pM$ and $U \in NM_p$, then $\omega(U, \Phi(X)) = \omega(\Xi(U), X)$, $\omega(U, J_\omega X) = \omega(J^\perp U, X)$, $g(U, \Phi(X)) = -g(\Xi(U), X)$.

We denote by ∇ both Levi-Civita connections of M and N or $F^{-1}TN$, if no confusion exists, otherwise we explicit them by ∇^M and ∇^N . We take on NM the usual connection ∇^\perp , given by $\nabla_X^\perp U = (\nabla_X U)^\perp$, for X and U smooth sections of TM and $NM \subset F^{-1}TN$, respectively. We denote the corresponding curvature tensors by R^M , R^N and R^\perp . The sign convention we choose for the curvature tensors is $R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X, Y]}Z$. The second fundamental form of F , $\nabla_X dF(Y) = \nabla dF(X, Y)$ is a symmetric 2-tensor on M that takes values on the normal bundle. Its covariant derivative $\nabla^\perp \nabla dF$ is defined considering ∇dF with

values on NM . We denote by $i_{NM} : NM \rightarrow F^{-1}TN$ the inclusion bundle map, and its covariant derivative $\nabla_X i_{NM}$ is a morphism from NM into TM . Then $\forall X, Y \in T_p M$, $U \in NM_p$, $p \in M$,

$$g(\nabla_X i_{NM}(U), Y) = g((\nabla_X^N U)^\top, Y) = -g(U, \nabla_X dF(Y)) = -g(A^U(X), Y) = -g(U, (\nabla_X^N Y)^\perp) \quad (2.5)$$

where $A : NM_p \rightarrow L(T_p M; T_p M)$ is the shape operator. Let $H = \frac{1}{\dim(M)} \text{trace}_{g_M} \nabla dF$ denote the mean curvature of F . F is *minimal* (resp. with parallel mean curvature) if $H = 0$ (resp. $\nabla^\perp H = 0$). F is J_ω -pluriminimal in \mathcal{L}_0 if $(\nabla dF)^{(1,1)}(X, Y) = \frac{1}{2}(\nabla dF(X, Y) + \nabla dF(J_\omega X, J_\omega Y)) = 0$. In this case F is minimal on \mathcal{L}_0 . For $p \in M$, $X, Y, Z \in T_p M$, $U, V \in NM_p$,

$$\nabla_Z F^* \omega(X, Y) = -g(\nabla_Z dF(X), \Phi(Y)) + g(\nabla_Z dF(Y), \Phi(X)) \quad (2.6)$$

$$\nabla_Z \omega^\perp(U, V) = -g(\nabla_Z i_{NM}(U), \Xi(V)) + g(\nabla_Z i_{NM}(V), \Xi(U)). \quad (2.7)$$

If (E, g_E) is a Riemannian vector bundle and $T, S : TM \rightarrow E$ are vector bundle maps, we define a 2-form $\langle T \wedge S \rangle$ by

$$\langle T \wedge S \rangle(X, Y) = g_E(T(X), S(Y)) - g_E(T(Y), S(X)).$$

From the symmetry of ∇dF , $\langle \nabla_Z dF \wedge \nabla_W dF \rangle(X, Y) = \langle \nabla_X dF \wedge \nabla_Y dF \rangle(Z, W)$. Recall the Gauss, Ricci and Coddazzi equations: For $X, Y, Z \in C^\infty(TM)$, and $U, V \in C^\infty(NM)$

$$R^M(X, Y, Z, W) = R^N(X, Y, Z, W) + \langle \nabla_Z dF \wedge \nabla_W dF \rangle(X, Y) \quad (2.8)$$

$$R^\perp(X, Y, U, V) = R^N(X, Y, U, V) + \langle A^U \wedge A^V \rangle(X, Y) \quad (2.9)$$

$$-R^N(X, Y, Z, U) = g(\nabla_X^\perp \nabla dF(Y, Z) - \nabla_Y^\perp \nabla dF(X, Z), U). \quad (2.10)$$

2.1 $\Delta\Phi, \Delta F^* \omega$

Lemma 2.1. *Let $F : M \rightarrow N$ be a $2m$ -dimensional immersed submanifold. For any $X, Y \in T_p M$, $U, V \in NM_p$, and any local o.n. frame e_i of M ,*

$$(i) \quad \nabla_X \Phi(Y) = \omega^\perp(\nabla_X dF(Y)) - \nabla_X dF((F^* \omega)^\sharp(Y)).$$

$$(ii) \quad d\Phi(X, Y) = -\nabla_X dF((F^* \omega)^\sharp(Y)) + \nabla_Y dF((F^* \omega)^\sharp(X)).$$

$$(iii) \quad \delta\Phi = -2m(JH)^\perp.$$

$$(iv) \quad \Delta\Phi(X) = 2m \left(\nabla_{(F^* \omega)^\sharp(X)}^\perp H - \nabla_X^\perp \omega^\perp(H) - \omega^\perp(\nabla_X^\perp H) \right) + \nabla_X dF(\delta(F^* \omega)^\sharp) + \\ + \sum_i \nabla dF(\nabla_{e_i} (F^* \omega)^\sharp(X), e_i) + \sum_i \left(R^N(e_i, X, (F^* \omega)^\sharp(e_i)) - R^N(e_i, (F^* \omega)^\sharp(X), e_i) \right)^\perp.$$

$$(v) \quad \Delta F^* \omega(X, Y) = 2m \left(g(\nabla_X^\perp H, \Phi(Y)) - g(\nabla_Y^\perp H, \Phi(X)) \right) + 2m g(H, d\Phi(X, Y)) \\ + \text{Trace}_M R^N(X, Y, dF(\cdot), \Phi(\cdot)) + \langle \nabla_X dF, \nabla_Y \Phi \rangle - \langle \nabla_Y dF, \nabla_X \Phi \rangle.$$

$$(vi) \quad g(\nabla_X \Xi(U), Y) = g((F^* \omega)^\sharp(\nabla_X i_{NM}(U)) - \nabla_X i_{NM}(\omega^\perp(U)), Y) = -g(\nabla_X \Phi(Y), U).$$

Proof. We take smooth vector fields X, Y of M such that at a given point p_0 , $\nabla Y(p_0) = \nabla X(p_0) = 0$, and assume also that $\nabla e_i = 0$ at p_0 . Then at p_0

$$\begin{aligned} \nabla_X \Phi(Y) &= \nabla_X^\perp(\Phi(Y)) = (\nabla_X(JdF(Y))^\perp)^\perp = (\nabla_X(JdF(Y) - (F^* \omega)^\sharp(Y)))^\perp \\ &= (J\nabla_X dF(Y))^\perp - \nabla_X dF((F^* \omega)^\sharp(Y)) = \omega^\perp(\nabla_X dF(Y)) - \nabla_X dF((F^* \omega)^\sharp(Y)) \end{aligned}$$

and we get (i). (ii) follows from $d\Phi(X, Y) = \nabla_X \Phi(Y) - \nabla_Y \Phi(X)$, and the symmetry of ∇dF . It follows that $\delta\Phi = -\sum_i \nabla_{e_i} \Phi(e_i) = -2m(JH)^\perp + \sum_i \nabla_{e_i} dF((F^*\omega)^\sharp(e_i))$. Note that $\sum_i \nabla_{e_i} dF((F^*\omega)^\sharp(e_i)) = 0$ because $\nabla dF(X, Y)$ is symmetric and $(F^*\omega)^\sharp$ is skew symmetric. Then (iii) is proved. Now,

$$\begin{aligned}
d\delta\Phi(X) &= -2m d((JH)^\perp)(X) = -2m \nabla_X^\perp(\omega^\perp(H)) = -2m \nabla_X^\perp \omega^\perp(H) - 2m \omega^\perp(\nabla_X^\perp H). \\
\delta d\Phi(X) &= -\sum_i \nabla_{e_i} d\Phi(e_i, X) = -\sum_i \nabla_{e_i}^\perp(d\Phi(e_i, X)) \\
&= -\sum_i \nabla_{e_i}^\perp(-\nabla_{e_i} dF((F^*\omega)^\sharp(X)) + \nabla_X dF((F^*\omega)^\sharp(e_i))) \\
&= \sum_i \nabla_{e_i}^\perp(\nabla_{(F^*\omega)^\sharp(X)} dF(e_i) - \nabla_X dF((F^*\omega)^\sharp(e_i))) \\
&= \sum_i \nabla_{e_i}^\perp \nabla dF((F^*\omega)^\sharp(X), e_i) + \nabla dF(\nabla_{e_i} (F^*\omega)^\sharp(X), e_i) \\
&\quad - \nabla_{e_i}^\perp \nabla dF(X, (F^*\omega)^\sharp(e_i)) - \nabla_X dF(\nabla_{e_i} ((F^*\omega)^\sharp)(e_i)) \\
&= \sum_i \nabla_{(F^*\omega)^\sharp(X)}^\perp \nabla dF(e_i, e_i) - (R^N(e_i, (F^*\omega)^\sharp(X))e_i)^\perp - \nabla_X^\perp \nabla dF(e_i, (F^*\omega)^\sharp(e_i)) \\
&\quad + (R^N(e_i, X)(F^*\omega)^\sharp(e_i))^\perp + \nabla dF(\nabla_{e_i} (F^*\omega)^\sharp(X), e_i) + \nabla_X dF(\delta((F^*\omega)^\sharp))
\end{aligned}$$

where we applied Coddazzi's equation (2.10) in the last equality. Since $\nabla_X^\perp \nabla dF$ is symmetric and $F^*\omega$ is skew symmetric $\sum_i \nabla_X^\perp \nabla dF(e_i, (F^*\omega)^\sharp(e_i)) = 0$. Thus,

$$\begin{aligned}
\delta d\Phi(X) &= \nabla_{(F^*\omega)^\sharp(X)}^\perp(2mH) + \sum_i (R^N(e_i, X)(F^*\omega)^\sharp(e_i) - R^N(e_i, (F^*\omega)^\sharp(X))e_i)^\perp \\
&\quad + \sum_i \nabla dF(\nabla_{e_i} (F^*\omega)^\sharp(X), e_i) + \nabla_X dF(\delta((F^*\omega)^\sharp)).
\end{aligned}$$

From $\Delta\Phi = (d\delta + \delta d)\Phi(X)$, we get the expression in (iv). Since $F^*\omega$ is closed, then, for Y vector field with $\nabla Y(p_0) = 0$, and using (2.10) and (2.6)

$$\begin{aligned}
\Delta F^*\omega(X, Y) &= (d\delta + \delta d)F^*\omega(X, Y) = d(\delta F^*\omega)(X, Y) = \\
&= \nabla_X(\delta F^*\omega)(Y) - \nabla_Y(\delta F^*\omega)(X) = \sum_i d(-\nabla_{e_i} F^*\omega(e_i, Y))(X) + d(\nabla_{e_i} F^*\omega(e_i, X))(Y) \\
&= \sum_i d(g(\nabla dF(e_i, e_i), \Phi(Y)) - g(\nabla dF(e_i, Y), \Phi(e_i)))(X) \\
&\quad + \sum_i d(g(\nabla dF(e_i, e_i), \Phi(X)) + g(\nabla dF(e_i, X), \Phi(e_i)))(X) \\
&= d(g(2mH, \Phi(Y)))(X) - d(g(2mH, \Phi(X)))(Y) - \sum_i g(\nabla_X^\perp \nabla dF(e_i, Y), \Phi(e_i)) \\
&\quad - \sum_i g(\nabla dF(e_i, Y), \nabla_X \Phi(e_i)) + g(\nabla_Y^\perp \nabla dF(e_i, X), \Phi(e_i)) + g(\nabla dF(e_i, X), \nabla_Y \Phi(e_i)) \\
&= 2m(g(\nabla_X^\perp H, \Phi(Y)) - g(\nabla_Y^\perp H, \Phi(X))) + 2m g(H, \nabla_X \Phi(Y)) - 2m g(H, \nabla_Y \Phi(X)) \\
&\quad + \sum_i g(-\nabla_X^\perp \nabla dF(Y, e_i) + \nabla_Y^\perp \nabla dF(X, e_i), \Phi(e_i)) \\
&\quad + \sum_i -g(\nabla dF(e_i, Y), \nabla_X \Phi(e_i)) + g(\nabla dF(e_i, X), \nabla_Y \Phi(e_i)) \\
&= 2m(g(\nabla_X^\perp H, \Phi(Y)) - g(\nabla_Y^\perp H, \Phi(X)) + g(H, d\Phi(X, Y))) + \sum_i R^N(X, Y, e_i, \Phi(e_i)) \\
&\quad + \sum_i -g(\nabla dF(e_i, Y), \nabla_X \Phi(e_i)) + g(\nabla dF(e_i, X), \nabla_Y \Phi(e_i))
\end{aligned}$$

obtaining the expression $\Delta F^*\omega$ of (v). Finally we prove (vi). From $\Xi(U) + \omega^\perp(U) = JU$, and assuming at a point p_0 $\nabla^\perp U(p_0) = 0$ (and so $\nabla_X^N U = \nabla_X i_{NM}(U)$) we obtain, at p_0

$$\begin{aligned}
\nabla_X \Xi(U) + \nabla_X i_{NM}(\omega^\perp(U)) &= (\nabla_X^N(\Xi(U) + \omega^\perp(U)))^\top = (J\nabla_X^N U)^\top \\
&= (J(\nabla_X i_{NM}(U)))^\top = (F^*\omega)^\sharp(\nabla_X i_{NM}(U)).
\end{aligned}$$

Therefore, using (2.5), and (i),

$$\begin{aligned} g(\nabla_X \Xi(U), Y) &= -g(\nabla_X i_{NM}(U), (F^* \omega)^\sharp(Y)) + g(\omega^\perp(U), \nabla_X dF(Y)) \\ &= g(\nabla_X dF((F^* \omega)^\sharp(Y)), U) - g(\omega^\perp(\nabla_X dF(Y)), U) = -g(\nabla_X \Phi(Y), U). \quad \square \end{aligned}$$

3 Cayley submanifolds

We assume $m = n$ and that $F : M^{2n} \rightarrow N^{2n}$ has equal Kähler angles (e.k.a.s), that is $\theta_\alpha = \theta \forall \alpha$ and we denote by $\mathcal{L} = \mathcal{L}_n$. In this case Φ and Ξ are conformal bundle maps and

$$(F^* \omega)^\sharp = \cos \theta J_\omega \quad \omega^\perp = \cos \theta J^\perp \quad (3.1)$$

$$-\Xi \circ \Phi = \sin^2 \theta Id_{TM} \quad -\Phi \circ \Xi = \sin^2 \theta Id_{NM}. \quad \text{on } M. \quad (3.2)$$

On $M \sim \mathcal{L}$, J_ω is g_M -orthogonal. Thus $\forall \alpha, \beta$,

$$g(\nabla_Z J_\omega(\alpha), \beta) = 2ig(\nabla_Z \alpha, \beta) = -g(\alpha, \nabla_Z J_\omega(\beta)), \quad g(\nabla_Z J_\omega(\alpha), \bar{\beta}) = 0. \quad (3.3)$$

The Ricci tensor of N can be expressed in terms of the frame (2.1) as (see [23])

$$\sin^2 \theta Ricci^N(U, V) = \sum_\alpha 4R^N(U, JV, \alpha, \Phi(\bar{\alpha})) = Trace_M R^N(U, JV, dF(\cdot), \Phi(\cdot)) \quad (3.4)$$

valid at all points $p \in M$, and $U, V \in T_{F(p)}N$. We have

Proposition 3.1. *Assume (N, J, g) is KE with Ricci = Rg, and $F : M \rightarrow N$ is a $2n$ -dimensional immersed submanifold with e.k.a.s. Then*

$$(1) \quad d\Phi(X, Y) = 2 \cos \theta (\nabla dF)^{(1,1)}(J_\omega X, Y) \text{ and } Trace_{\mathbb{C}, J_\omega} d\Phi = \sum_\alpha d\Phi(X_\alpha, Y_\alpha) = 2n \cos \theta H.$$

$$(2) \quad d \sin^2 \theta(X) \cdot g(Y, Z) = g(\nabla_X \Phi(Y), \Phi(Z)) + g(\nabla_X \Phi(Z), \Phi(Y)).$$

(3) $d\Phi = 0$ iff $d\Phi(X, J_\omega X) = 0$ iff F is J_ω -pluriminimal or Lagrangian. Furthermore: (a) if $R \neq 0$, then $d\Phi = 0$ iff F is complex or Lagrangian; (b) if $R = 0$, $d\Phi = 0$ iff F has constant Kähler angle and $\Phi : (TM, \nabla, g_M) \rightarrow (NM, \nabla^\perp, g)$ is a parallel homothetic morphism.

(4) $\delta\Phi = 0$ iff H is a Lagrangian direction of NM , iff F is minimal away from \mathcal{L} . Consequently, $\Phi : (TM, \nabla, g_M) \rightarrow (NM, \nabla^\perp)$ is closed and co-closed 1-form (and so harmonic) iff $\Phi : (TM, \nabla, g_M) \rightarrow (NM, \nabla^\perp)$ is parallel iff F is Lagrangian or J_ω - pluriminimal.

(5) If F has parallel mean curvature then

$$\Delta\Phi(X) = -2n \nabla_X^\perp \omega^\perp(H) + \nabla_X dF(\delta(F^* \omega)^\sharp) + \sum_i \nabla dF(\nabla_{e_i}(F^* \omega)^\sharp(X), e_i) \quad (3.5)$$

$$+ \sum_i (R^N(e_i, X, (F^* \omega)^\sharp(e_i)) - R^N(e_i, (F^* \omega)^\sharp(X), e_i))^\perp \quad (3.6)$$

$$\begin{aligned} \Delta F^* \omega(X, Y) &= 2n g(H, -\nabla_X dF((F^* \omega)^\sharp(Y)) + \nabla_Y dF((F^* \omega)^\sharp(X))) \\ &\quad + \sin^2 \theta R F^* \omega(X, Y) + \sum_i 2\omega^\perp(\nabla_{e_i} dF(Y), \nabla_{e_i} dF(X)) \\ &\quad + \langle \nabla_Y dF, \nabla_X dF \circ (F^* \omega)^\sharp \rangle - \langle \nabla_X dF, \nabla_Y dF \circ (F^* \omega)^\sharp \rangle. \end{aligned}$$

Proof. (1) follows from Lemma 2.1, (2) from differentiation of (2.2), (3) and (4) are consequence of [24] and Lemma 2.1 (ii) and (iii), and (5) follows directly from Lemma 2.1., (3.4) and the

J -invariance of Ricci^N . □

Now we assume $n = 2$. Four dimensional submanifolds of any Kähler manifold of complex dimension ≥ 4 , immersed with equal Kähler angles, are just the same as submanifolds satisfying

$$*F^*\omega = \pm F^*\omega.$$

Since pointwise $F^*\omega$ is self-dual or anti-self-dual, and is a closed 2-form, then it is co-closed as well. In particular it is an harmonic 2-form. This is not the case of $n \neq 2$, unless if $\theta = \text{constant}$ (see [24], or next lemma 3.1(2)). In case that N is a KE manifold of zero Ricci tensor and of real dimension 8, a Cayley submanifold is a minimal 4-dimensional submanifold with equal Kähler angles $\theta_1 = \theta_2 = \theta$. If N is a Calabi-Yau 4-fold, that is a Kähler manifold with a complex volume form $\rho \in \wedge_{\mathbb{C}}^{(4,0)} M$ (this condition implies Ricci-flat, and the converse also holds in case N is simply connected), these submanifolds are characterised by being calibrated by one of the Cayley calibrations $\Omega = \frac{1}{2}\omega \wedge \omega + \text{Re}(\rho)$, where ρ is one of the S^1 -family of parallel holomorphic volumes of N ([12]). Calabi-Yau 4-folds are $\text{Spin}(7)$ manifolds. So, locally on N there is a section $\{e_1, \dots, e_8\}$ of the principal $\text{Spin}(7)$ -bundle of frames of N defined on a open set U of N , and that at each point $p \in U$, defines a isometry of $T_p N$ onto \mathbb{R}^8 such that Ω looks like (see [16])

$$\begin{aligned} \Omega = & dx_{1234} + dx_{5678} + (dx_{12} + dx_{34}) \wedge (dx_{56} + dx_{78}) \\ & + (dx_{13} - dx_{24}) \wedge (dx_{57} - dx_{68}) - (dx_{14} + dx_{23}) \wedge (dx_{58} + dx_{67}). \end{aligned} \quad (3.7)$$

From this equation we see that the subspaces spanned by e_1, \dots, e_4 and e_5, \dots, e_8 are Cayley subspaces. We note that we use the opposite orientation on Cayley subspaces that Harvey and Lawson do in [12], and the calibration they use is given by $\Omega' = -\frac{1}{2}\omega^2 + \text{Re}(\rho)$ that is

$$\begin{aligned} \Omega' = & dx_{1234} + dx_{5678} + (dx_{12} - dx_{34}) \wedge (dx_{56} - dx_{78}) \\ & + (dx_{13} + dx_{24}) \wedge (dx_{57} + dx_{68}) + (dx_{14} - dx_{23}) \wedge (dx_{58} - dx_{67}) \end{aligned} \quad (3.8)$$

and so $-\Omega'$ and Ω differ on the chosen parallel holomorphic volume, giving opposite phase on the special Lagrangian calibration. In [12] it is proved that $\text{Spin}(7)$ acts transitively on the grassmannian $G(\Omega)$ of Cayley 4-planes of \mathbb{R}^8 and the isotropic subgroup of a Cayley subspace E is $K \equiv SU(2) \times SU(2) \times SU(2)/\mathbb{Z}_2$ (see in (1.39) of [12] how K is embedded in $\text{Spin}(7)$). Thus, we can assume that $B = \{e_1, e_2, e_3, e_4\}$ and $B^\perp = \{e_5, e_6, e_7, e_8\}$ are direct o.n. basis of $T_p M$ and NM_p respectively. We identify isometrically in the usual way bivectors with 2-forms. So $J_1^B = e_1 \wedge e_2 + e_3 \wedge e_4$, $J_2^B = e_1 \wedge e_3 - e_2 \wedge e_4$, and $J_3^B = e_1 \wedge e_4 + e_2 \wedge e_3$ defines a direct o.n. basis (of norm $\sqrt{2}$) of $\wedge_+^2 T_p M$. Similar for $\wedge_+^2 NM_p$. We define a bilinear map:

$$\begin{aligned} \Omega^\Delta : & \quad \wedge_+^2 T_p M \times \wedge_+^2 NM_p \rightarrow \mathbb{R} \\ & \quad \Omega^\Delta(X \wedge Y, U \wedge V) = \Omega(X, Y, U, V) \end{aligned}$$

Proposition 3.2. Ω^Δ defines a natural orientation reversing isometric bundle isomorphism between $\wedge_+^2 TM$ and $\wedge_+^2 NM$.

Proof. Identifying isometrically in the canonic way (via musical isomorphisms with respect to the induced metrics) the bilinear map Ω^Δ restricted to $\wedge_+^2 T_p M \times \wedge_+^2 NM_p$ with a linear map $\Omega^\Delta : \wedge_+^2 T_p M \rightarrow \wedge_+^2 NM_p$, and using the frame e_i adapted to M , from (3.7) we see that Ω^Δ

applies J_1^B to $J_1^{B^\perp}$, J_2^B to $J_2^{B^\perp}$ and J_3^B to $-J_3^{B^\perp}$, and so it gives a global orientation reversing isometry bundle map between the vector bundles $\bigwedge_+^2 TM$ and $\bigwedge_+^2 NM$. \square

In particular, $p_1(\bigwedge_+^2 TM) = p_1(\bigwedge_+^2 NM)$. We will see in section 5 that this equality still holds for the case of M Cayley but N only a Ricci-flat Kähler manifold. In this case, we cannot guarantee the existence of a global isomorphism between the two bundles. If M were calibrated for Ω' we would obtain a global orientation preserving isometry bundle map between the vector bundles $\bigwedge_-^2 TM$ and $\bigwedge_-^2 NM$.

For simplicity of notation we denote by

$$g_Z XY = g(\nabla dF(Z, X), JdF(Y)) = g(\nabla_Z dF(X), \Phi(Y)) \quad (3.9)$$

and define $\omega_M(X, Y) = g_M(J_\omega X, Y)$. We have

$$d\omega_M(X, Y, Z) = g(\nabla_X J_\omega(Y), Z) - g(\nabla_Y J_\omega(X), Z) + g(\nabla_Z J_\omega(X), Y) \quad (3.10)$$

$$= g(dJ_\omega(X, Y), Z) + g(\nabla_Z J_\omega(X), Y). \quad (3.11)$$

Recall the Weitzenböck operator of $\bigwedge^2 T^*M$ applied to $F^*\omega$ is given by

$$SF^*\omega(X, Y) = \sum_i -\bar{R}(e_i, X)F^*\omega(e_i, Y) + \bar{R}(e_i, Y)F^*\omega(e_i, X)$$

where e_i is an o.n.b. of $T_p M$, and \bar{R} is the curvature operator on $\bigwedge^2 T^*M$: $\forall X, Y, u, v \in T_p M, \phi \in \bigwedge^2 T_p^* M, (\bar{R}(X, Y)\phi)(u, v) = -\phi(R^M(X, Y)u, v) - \phi(u, R^M(X, Y)v)$. Let $s^M = \text{trace Ricci}^M = \sum_\mu 4\text{Ricci}^M(\mu, \bar{\mu})$ be the scalar curvature of M .

Lemma 3.1. *For an immersion with e.k.a.s, $\forall X \in T_p M, \forall \alpha, \beta$:*

- (1) $\|\nabla F^*\omega\|^2 = n\|\nabla \cos \theta\|^2 + \frac{1}{2}\cos^2 \theta \|\nabla J_\omega\|^2$.
- (2) $\delta((F^*\omega)^\sharp) = (n-2)J_\omega(\nabla \cos \theta)$.
- (3) $\cos \theta(\delta J_\omega) = (n-1)J_\omega(\nabla \cos \theta)$.
- (4) $\delta(F^*\omega)(X) = \sum_\mu (-2g_X \mu \bar{\mu} - 2g_X \bar{\mu} \mu) + 2ng(H, JdF(X))$.
- (5) $g_X \beta \alpha = g_X \alpha \beta + \cos \theta g(\nabla_X J_\omega(\alpha), \beta)$.
- (6) $g_X \bar{\beta} \alpha = g_X \alpha \bar{\beta} + \frac{i}{2} d \cos \theta(X)$.
- (7) $\frac{i}{2} d \cos \theta(X) = -g_X \beta \bar{\beta} + g_X \bar{\beta} \beta$ (no summation on β).
- (8) For $n=2$, $\delta F^*\omega = 0$, and if $H=0$ $\frac{i}{2} d \cos \theta(X) = \sum_\mu -g_X \mu \bar{\mu} = \sum_\mu g_X \bar{\mu} \mu$.
- (9) $S^M = \sum_{\mu\rho} 8R^M(\mu, \rho, \bar{\mu}, \bar{\rho}) + 8R^M(\mu, \bar{\rho}, \bar{\mu}, \rho) = \sum_{\mu\rho} 16R^M(\mu, \rho, \bar{\mu}, \bar{\rho}) - 8R^M(\mu, \bar{\mu}, \rho, \bar{\rho})$.
- (10) $\langle SF^*\omega, F^*\omega \rangle = 16 \cos^2 \theta \sum_{\mu\rho} R^M(\mu, \rho, \bar{\mu}, \bar{\rho}) = \cos^2 \theta s^M + \sum_{\mu\rho} 8 \cos^2 \theta R^M(\mu, \bar{\mu}, \rho, \bar{\rho})$.
- (11) For $n=2$, $H=0$, and N Ricci-flat, $\Delta \cos^2 \theta = \langle SF^*\omega, F^*\omega \rangle + \|\nabla F^*\omega\|^2$.

Proof. All formulas are somewhere proved in [24]. We only need to check (4) and part of (8). Since $2nH = \sum_\mu 2\nabla_\mu dF \bar{\mu} + \sum_\mu 2\nabla_{\bar{\mu}} dF \mu$ and ∇dF is symmetric, by (2.6)

$$\begin{aligned} \delta F^*\omega(X) &= \sum_\mu -2\nabla_\mu F^*\omega(\bar{\mu}, X) - 2\nabla_{\bar{\mu}} F^*\omega(\mu, X) = \sum_\mu 2g_\mu \bar{\mu} X - 2g_\mu X \bar{\mu} + 2g_{\bar{\mu}} \mu X - 2g_{\bar{\mu}} X \mu \\ &= g(2nH, JdF(X)) - \sum_\mu (2g_\mu X \bar{\mu} + 2g_{\bar{\mu}} X \mu). \end{aligned}$$

For $n = 2$, from (2), $\delta F^* \omega = 0$ and so if F is minimal by (4), $\sum_{\mu} g_X \mu \bar{\mu} = -\sum_{\mu} g_X \bar{\mu} \mu$. Finally, by (7) $id \cos \theta(X) = \sum_{\mu} -g_X \mu \bar{\mu} + g_X \bar{\mu} \mu = \sum_{\mu} -2g_X \mu \bar{\mu} = \sum_{\mu} 2g_X \bar{\mu} \mu$. \square

Consequently

Proposition 3.3. *If $n = 2$ and $F : M \rightarrow N$ is a submanifold with e.k.a.s, then*

- (1) $d\omega_M = -d \log \cos \theta \wedge \omega_M$,
- (2) $d \cos \theta(1) = -2i \cos \theta g(\nabla_{\bar{2}} J_{\omega}(2), 1) = 4 \cos \theta g(\nabla_{\bar{2}} 2, 1)$,
 $d \cos \theta(2) = -2i \cos \theta g(\nabla_{\bar{1}} J_{\omega}(1), 2) = 4 \cos \theta g(\nabla_{\bar{1}} 1, 2)$,
- (3) *and if F is a Cayley submanifold, then $g_X 2\bar{2} = -g_X \bar{1}1$.*

Proof. (1) From $0 = dF^* \omega = d(\cos \theta \omega_M) = d \cos \theta \wedge \omega_M + \cos \theta d\omega_M$ we obtain (1). From (1), $\frac{i}{2} d \cos \theta(1) = d \cos \theta \wedge \omega_M(1, 2, \bar{2}) = -\cos \theta d\omega_M(1, 2, \bar{2})$. Equations (3.3) and (3.10) give $d\omega_M(1, 2, \bar{2}) = g(\nabla_{\bar{2}} J_{\omega}(1), 2) = -g(\nabla_{\bar{2}} J_{\omega}(2), 1) = -2ig(\nabla_{\bar{2}} 2, 1)$ obtaining the first equality in (2). Similar for $d \cos \theta(2)$. Lemma 3.1(7)(8), and minimality of F imply $-2g_X 1\bar{1} + 2g_X \bar{1}1 = id \cos \theta(X) = \sum_{\mu} -2g_X \mu \bar{\mu} = -2g_X 1\bar{1} - 2g_X 2\bar{2}$ and we get (3). \square

3.1 The almost complex structure J_{ω}

Proposition 3.4. *If $n = 2$, and F has e.k.a.s, on $M \sim \mathcal{L}$, we have*

- (1) $dJ_{\omega} = 0$ iff $\nabla J_{\omega} = 0$. In this case $d\omega_M = 0$.
- (2) $\cos \theta = \text{const.}$ iff $d\omega_M = 0$ iff $\nabla_{\bar{\gamma}} J_{\omega}(\alpha) = 0$ iff $\delta J_{\omega} = 0$.
- (3) If $\nabla J_{\omega} = 0$ then $\cos \theta = \text{constant}$ and $F^* \omega$ parallel.
- (4) J_{ω} is integrable iff $dJ_{\omega}(1, 2) = 0$ iff $\nabla_{\gamma} J_{\omega}(\alpha) = 0$.
- (5) If F is J_{ω} -pluriminimal then $\cos \theta = \text{constant}$.

Consequently we have:

- (A) If J_{ω} is integrable, then $\cos \theta = \text{constant}$ iff J_{ω} is Kähler.
- (B) If J_{ω} is an almost complex structure of the Gray list [10] that is, Kähler, or almost, quasi, nearly or semi-Kähler, then $\cos \theta$ is constant. Therefore, immersions with non-constant e.k.a.s produces submanifolds with an almost complex structure on $M \sim \mathcal{L}$, that might be integrable but not one of the Gray list.

Proof. From (3.10) $dJ_{\omega} = 0$ implies

$$d\omega_M(X, Y, Z) = g(\nabla_Z J_{\omega}(X), Y) \quad (3.12)$$

with $(Z, X) \rightarrow \nabla_Z J_{\omega}(X)$ symmetric. But then $d\omega_M = 0$. By (3.12) we get $\nabla J_{\omega} = 0$, and we have proved (1). Since $\nabla J_{\omega}(T^{1,0}M) \subset T^{0,1}M$ (see (3.3)), then, $d\omega_M(\alpha, \beta, \bar{\gamma}) = g(\nabla_{\bar{\gamma}} J_{\omega}(\alpha), \beta)$, and for $n = 2$, α, β, γ must have repeated vectors, so, $0 = d\omega_M(\alpha, \beta, \gamma) = g(dJ_{\omega}(\alpha, \beta), \gamma) + g(\nabla_{\gamma} J_{\omega}(\alpha), \beta)$. Using Prop.3.3(2) and (3.3)

$$\begin{aligned} \|d\omega_M\|^2 &= \sum_{\gamma, \alpha < \beta} 16|d\omega_M(\alpha, \beta, \bar{\gamma})|^2 = \sum_{\gamma, \alpha < \beta} 16|\langle \nabla_{\bar{\gamma}} J_{\omega}(\alpha), \beta \rangle|^2 = \sum_{\gamma, \alpha} 8\|\nabla_{\bar{\gamma}} J_{\omega}(\alpha)\|^2 \\ &= \frac{4}{\cos^2 \theta} (|d \cos \theta(1)|^2 + |d \cos \theta(2)|^2) = 2\|\nabla \log(\cos \theta)\|^2 \end{aligned} \quad (3.13)$$

and we have proved the first 3 equivalences of (2). The last one comes from lemma 3.1(3). If $\nabla J_\omega = 0$ then $\delta J_\omega = 0$. Thus, by (2) $\cos \theta$ is constant, and by Lemma 3.1(1) $\nabla F^* \omega = 0$ and (3) is proved. The integrability of J_ω is equivalently to the vanishing of the tensor: $N_{J_\omega}(X, Y) = [J_\omega X, J_\omega Y] - [X, Y] - J_\omega[X, J_\omega Y] - J_\omega[J_\omega X, Y]$. Using the connection on M we have $N_{J_\omega}(X, Y) = -J_\omega(dJ_\omega(X, Y)) + (\nabla_{J_\omega X} J_\omega)(Y) - (\nabla_{J_\omega Y} J_\omega)(X)$. From $J_\omega \circ J_\omega = -Id$ we have $\nabla_X J_\omega(J_\omega Y) = -J_\omega(\nabla_X J_\omega(Y))$. Thus, $N_{J_\omega} = 0$ iff $dJ_\omega(X, Y) = dJ_\omega(J_\omega X, J_\omega Y)$, iff $dJ_\omega(\alpha, \beta) = 0$, iff $\nabla_\gamma J_\omega(\alpha) = 0$. Recall that pluriminimality implies $\cos \theta = \text{constant}$ ([24]). Now we prove (A). If J_ω is integrable and $\cos \theta$ is constant, by (2) and (4) $\nabla J_\omega = 0$, i.e. J_ω is Kähler. Now we prove the last remark (B). If J_ω is almost-Kähler, that is $d\omega_M = 0$, from (2) $\cos \theta = \text{constant}$. If J_ω is nearly-Kähler, that is $\nabla_X J_\omega(X) = 0$, then $\delta J_\omega = 0$, and so it is semi-Kähler. The later implies by (2) that θ is constant. If J_ω is quasi-Kähler, then $\nabla_X J_\omega(Y) = -\nabla_{J_\omega X} J_\omega(J_\omega Y)$, and so $\nabla_1 J_\omega(\bar{1}) = \nabla_2 J_\omega(\bar{2}) = 0$. By Proposition 3.3 this implies $\cos \theta = \text{constant}$. \square

Remark. As an observation, we conclude that if $R \neq 0$ and M is compact immersed with e.k.as and with almost Kähler J_ω (i.e. $d\omega_M = 0$), then M is Kähler, confirming the Goldberg conjecture in this case. In fact, from previous proposition we have θ constant, and so theorem 1.2 of [20] concludes that J_ω is Kähler.

Let $B = \{X_1, Y_1, X_2, Y_2\}$ be a diagonalising o.n. basis of $F^* \omega$ at the point $p \in M$, and let $\epsilon(p) \in \{-1, +1\}$ be the sign of this basis (well defined for $p \notin \mathcal{L}$), and if $p \in M \sim \mathcal{C}$, we denote by $\epsilon'(p)$ the sign of the basis $B' = \Phi(B)$ of NM_p . Then $\frac{\Phi(Y_1)}{\sin \theta}, \frac{\Phi(X_1)}{\sin \theta}, \frac{\Phi(Y_2)}{\sin \theta}, \frac{\Phi(X_2)}{\sin \theta}$ diagonalizes ω^\perp , and has the same orientation as B' .

Lemma 3.2. *Let $n = 2$ and $F : M \rightarrow N$ be an immersed oriented submanifold with e.k.a.s. Then $\Phi : TM \rightarrow NM$ is, along $M \sim \mathcal{L} \cup \mathcal{C}$, an orientation preserving bundle morphism, with respect to the orientations defined by J_ω and J^\perp respectively. Moreover, for $p \in M \sim \mathcal{C}$, $\epsilon(p)\epsilon'(p) = +1$ always hold.*

Proof. From (2.3) Φ is a $J_\omega - J^\perp$ -anti-holomorphic morphism, and so it preserves the orientations. If $p \in \mathcal{L}$, for any chosen basis B , $\epsilon(p)\epsilon'(p) = +1$ still holds. Now assume $p \notin \mathcal{L} \cup \mathcal{C}$. Since $\Phi(X_\alpha) = JX_\alpha - \cos \theta Y_\alpha$, $\Phi(Y_\alpha) = JY_\alpha + \cos \theta X_\alpha$, then

$$\begin{aligned} 0 \neq A_\Phi(T_p M) &:= \text{Vol}_N(X_1, Y_1, X_2, Y_2, \Phi(X_1), \Phi(Y_1), \Phi(X_2), \Phi(Y_2)) = \epsilon(p)\epsilon'(p) \sin^4 \theta \\ &= \text{Vol}_N(X_1, Y_1, X_2, Y_2, JX_1, JY_1, JX_2, JY_2) = \text{Vol}_N(X_1, JX_1, Y_1, JY_1, X_2, JX_2, Y_2, JY_2). \end{aligned}$$

Note that if $T_p M$ is a Lagrangian subspace, then $A_\Phi = 1$ for any o.n.b. X_1, Y_1, X_2, Y_2 of $T_p M$ we choose. Now we prove that $A_\Phi > 0$ holds for any $T_p M$ with e.k.a.s with $\cos \theta \neq 1$. We consider the strictly decreasing continuous curve, $a : [0, \frac{\pi}{2}] \rightarrow [0, 1]$, defined by $a(t) = \frac{1 - \sin(t)}{\cos(t)}$, $a(0) = 1$, and $a(\frac{\pi}{2}) = 0$. We may identify $T_p N$ with $V = \mathbb{R}^4 \times \mathbb{R}^4$, with complex structure $J_0(X, Y) = (-Y, X)$. We consider the family of maps, for $t \in [0, \frac{\pi}{2}]$, $\Gamma_t : \mathbb{R}^4 \rightarrow V$, $\Gamma_t(X) = (X, a(\frac{\pi}{2} - t)J_\omega(X))$, where J_ω is a fixed g_0 -orthogonal complex structure on \mathbb{R}^4 . For each t , Γ_t is an isomorphism of \mathbb{R}^4 onto a subspace $E_t = \Gamma_t(\mathbb{R}^4)$ with e.k.a $\theta = (\frac{\pi}{2} - t)$ and complex structure J_ω (see [20]), giving a continuous curve of subspaces starting from a Lagrangian subspace $E_0 = \mathbb{R}^4$ of V . Then $t \in [0, \frac{\pi}{2}[\rightarrow A_{\Phi_t}(E_t)$ is a continuous curve of nonzero numbers, starting with

value 1 at $t = 0$. Thus it remains positive on $[0, \frac{\pi}{2}[$. Now, all subspaces E of dimension 4 of V that have the same Kähler angles are the same up to a unitary transformation of V , (such transformation maps a diagonalising basis of E and of E^\perp into the corresponding ones of E' and E'^\perp , see [20] or [26]). This proves $\epsilon(p)\epsilon'(p) = \frac{A_\Phi(T_p M)}{\sin^4 \theta} > 0$. \square

Recall that a hyper-Kähler manifold is a Riemannian manifold (N, g) endowed with two Kähler structures I, J that anti-commute, $IJ = -JI$. Then $I, J, K := IJ$ define a family of Kähler structures indexed on S^2 , $(J_x)_{x \in S^2}$, $J_x = aI + bJ + cK$, $x = (a, b, c)$.

Proposition 3.5. *Let $n = 2$, and $F : M \rightarrow N$ be an immersed connected oriented submanifold with e.k.as. Then we have:*

- (1) *With respect to the given orientation of M , $\Phi : TM \rightarrow NM$ is, away from \mathcal{C} , an orientation preserving morphism. Furthermore, we may assume that the orientation of M is such that $F^*\omega$ is self-dual on all M , and so J_ω and J^\perp define the orientation of M and NM resp..*
- (2) *If $(N, (J_x)_{x \in S^2}, g)$ is hyper-Kähler, and $F : M \rightarrow N$ is a J_x -complex submanifold, then $\forall y \in S^2$, $F^*\omega_y$ is selfdual, where ω_y is the Kähler form of (N, J_y, g) . Moreover M is a Cayley submanifold of (N, J_y, g) with k.a $\cos \theta_y(p) = \|(J_y X)^\top\|$ where $X \in T_p M$ is any unit vector.*

Proof. (1) follows immediately from Lemma 3.2. Each of the 2-forms $\eta_\pm = F^*\omega \pm *F^*\omega$ is harmonic and so, if not identically to zero, its zero set has empty interior. This implies that one of the η_\pm must vanish identically. Thus, we may choose the orientation of M s.t. $F^*\omega$ is self-dual on all M , and so J_ω defines the orientation of M . From Lemma 3.2 J^\perp defines the orientation of NM . Now we prove (2). First we recall that for any $x, y \in S^2$ $J_x J_y = J_{x \cdot y} = -\langle x, y \rangle Id + J_{x \times y}$. Let $B = \{e_1, e_2, e_3, e_4\} = \{X, J_x X, Z, J_x Z\}$ be an o.n.b. of $T_p M$. We have $g(J_y J_x X, X) = -\langle y, x \rangle = g(J_y J_x Z, Z)$, $g(J_y J_x Z, X) = -g(J_y X, J_x Z)$, $g(J_y Z, J_x X) = -g(J_{y \times x} X, Z)$, and $g(J_y J_x Z, J_x X) = g(J_y X, Z)$. A basis for the self-dual 2-forms on M is given by

$$J_i^B = e_1 \wedge e_2 + e_3 \wedge e_4, \quad J_j^B = e_1 \wedge e_3 - e_2 \wedge e_4, \quad J_k^B = e_1 \wedge e_4 + e_2 \wedge e_3. \quad (3.14)$$

Then we see that $F^*\omega_y = \cos \theta_y(p) J_u^B$ where $u = \frac{1}{t}(\langle y, x \rangle i + g(J_y X, Z)j + g(J_{y \times x} X, Z)k)$ (i, j, k is the usual basis of \mathbb{R}^3) and

$$\cos \theta_y(p) = t = \sqrt{(\langle y, x \rangle^2 + g(J_y X, Z)^2 + g(J_{y \times x} X, Z)^2)} = \|(J_y X)^\top\|,$$

proving that $F^*\omega_y$ is self-dual. In particular M is a Cayley submanifold (see also [21]). \square

3.2 A particular case

Let us first assume that N is an hyper-Kähler (HK) manifold (N, I, J, K, g) of complex dimension 4, where I and J are g -orthogonal Kähler structures on N that anti-commute and $K = IJ$. If M is an I -complex submanifold of complex dimension 2, then from Prop.3.5 M is a J -Cayley submanifold of N . Let ω_J be the Kähler form of (N, J, g) . Then $F^*\omega_J^\sharp = \cos \theta_J J_{\omega_J}$. Similar for K . We are going to describe an o.n. basis that diagonalizes $F^*\omega_J$ and ω_J^\perp that we use in [2] and [25]. Let $p \in M$ and $X \in T_p M$ be a unit vector and $H_X = \text{span}\{X, IX, JX, KX\}$. Since $T_p N$ is a vector space of dimension 8 and $X \in T_p M$, there exist $U \in H_X^\perp \cap NM_p$ unit vector and

o.n. basis B of $T_p M$ and B^\perp of NM_p of the form

$$\begin{aligned} B &= \{W_1, W_2 = IW_1, W_3, W_4 = IW_3\} = \{X, IX, J(cX + sU), K(cX + sU)\} \\ B^\perp &= \{U_1, U_2 = IU_1, U_3, U_4 = IU_3\} = \{-U, -IU, J(cU - sX), K(cU - sX)\} \end{aligned}$$

where $c^2 + s^2 = 1$. The basis $\{W_1, J_{\omega_J} W_1, W_2, J_{\omega_J} W_2\}$ and $\{W_1, J_{\omega_K} W_1, W_2, J_{\omega_K} W_2\}$ diagonalize $F^* \omega_J$ and $F^* \omega_K$ respectively with $\cos \theta_J = |c| = \cos \theta_K$. Moreover, $I = J_i^B$, $J_{\omega_J} = \epsilon J_j^B$, $J_{\omega_K} = \epsilon J_k^B$, where $\epsilon = \text{sign } c$ (if $c = 0$ take $\epsilon = 1$) (see (3.14)). Consequently, the hyper-Hermitian structure on TM defined by the orientation determined by I is given by $\{I, J_{\omega_J}, J_{\omega_K}\}$, defined on \mathcal{L}_0 , that is, away from totally complex points, i.e. points with $c=0$, (see next remark).

Remark. A convenient multiple of the m -power of the fundamental 4-form Ω gives a calibration ([27]). Then we may define a quaternionic angle for any $4m$ -dimensional submanifold. The Ω -angle of a complex 4-submanifold (in the sense of quaternionic-Kähler geometry) is given by $\theta(p)$ s.t. $\cos \theta(p) = \Omega(X_1, X_2, X_3, X_4)$ where X_i is a d.o.n.b. of $T_p M$, and has values between $\frac{1}{3}$ and 1. The first extreme value corresponds to a *totally complex point*, that is a point s.t. $T_p M$ is I -complex and J -Lagrangian, for a local almost hyper-Hermitian structure $I, J, K = IJ$ of N . The other extreme value corresponds to a *quaternionic point*, a point s.t. $T_p M$ is a quaternionic subspace of $T_p N$, or equivalently $T_p M$ is I and J -complex.

Now we return to the general case of (N, J, g) being a Ricci-flat KE 4-fold and $F : M \rightarrow N$ an immersed 4-submanifold with e.k.a.s. For each non J -Lagrangian point p and local almost complex structure I on M orthogonal to J_ω we can find a local basis $Z_{\alpha 1 \leq \alpha \leq 2}$ defined by (2.1) satisfying

$$I(1) = \bar{2} \quad I(2) = -\bar{1} \quad I(\bar{1}) = 2 \quad I(\bar{2}) = -1. \quad (3.15)$$

In this subsection we are going to assume that for each $p_0 \in M \sim \mathcal{L}$ a Kähler complex structure I orthogonal to J_ω exists on a open dense set O of a neighbourhood of p_0 , and we will say that M is *I -Kähler*. Then (see e.g. [9], [18]) we have the following orthogonal decomposition w.r.t. I on O :

$$\bigwedge_+^2 TM = \mathbb{R} \omega_I \oplus (T^{(2,0)} M \oplus T^{(0,2)} M) \quad (3.16)$$

where ω_I denotes the I -Kähler form on O . Moreover any real self-dual harmonic 2-form ζ orthogonal to ω_I (for the Hilbert-Schmidt inner product) is of the form $\zeta = \varphi + \bar{\varphi}$ where φ is a holomorphic $(2,0)$ -form over O . The I -Ricci form ρ^M , $\rho^M(X, Y) = \text{Ricci}^M(IX, Y)$ and the scalar curvature s^M of M is given by

$$\rho^M = -i\partial\bar{\partial} \log \|\varphi\|^2 = -i\partial\bar{\partial} \log \|\zeta\|^2 \quad (3.17)$$

$$s^M = \Delta^+ \log \|\varphi\|^2 = \Delta^+ \log \|\zeta\|^2 \quad (3.18)$$

away from the zero set of ζ . If we take $\zeta = F^* \omega_J$ we conclude:

Theorem 3.1. *on a open set O of M where M is I -Kähler*

$$\text{Ricci}^M(IX, Y) = -\frac{1}{2} d(d \log \cos^2 \theta \circ I)(X, Y) \quad (3.19)$$

$$s^M = \Delta \log \cos^2 \theta. \quad (3.20)$$

Proposition 3.6. *If M is I -Kähler, then J_ω is integrable iff θ is constant. In the particular case that N is hyper-Kähler, and M is I -complex submanifold, then J_ω is integrable iff θ is constant, iff J_ω is Kähler, iff $(M, I, J_\omega, IJ_\omega)$ is hyper-Kähler.*

proof. Since J_ω and I anti-commute and I is parallel on M , then $(\nabla J_\omega) \circ I = \nabla(J_\omega \circ I) = -\nabla(I \circ J_\omega) = -I \circ \nabla J_\omega$. Therefore, $g(\nabla_Z J_\omega(\bar{1}), \bar{2}) = g(\nabla_Z J_\omega(-I(2)), I(1)) = g(I \nabla_Z J_\omega(2), I(1)) = g(\nabla_Z J_\omega(2), 1)$. By Prop.3.3(2) and (3.15) $d \cos \theta(1) = 2i \cos \theta g(\nabla_{I(1)} J_\omega(1), 2)$, $d \cos \theta(2) = 2i \cos \theta g(\nabla_{I(2)} J_\omega(1), 2)$. Thus, $d \cos \theta(Z) = 2i \cos \theta g(\nabla_{IZ} J_\omega(1), 2)$, or equivalently

$$\nabla_Z J_\omega(Y) = -d \log \cos \theta(IZ) J_\omega I(Y). \quad (3.21)$$

From Prop.3.4(4) and the above formula (3.21) we conclude that J_ω is integrable iff $0 = g(dJ_\omega(1, 2), 1) = \frac{1}{2} d \log \cos \theta(1)$ and $0 = g(dJ_\omega(1, 2), 2) = -\frac{1}{2} d \log \cos \theta(2)$, that is, iff $\cos \theta$ is constant. This condition turns out to be equivalent to J_ω to be Kähler, by Prop.3.4(A). \square .

4 Complex and Lagrangian points

In this section we will study the nature of the complex and the Lagrangian points of a submanifold $F : M^{2n} \rightarrow N^{2n}$ immersed with parallel mean curvature and with e.k.as. We introduce some natural complex vector subbundles \mathcal{F}^+ and \mathcal{F}^- of $F^{-1}TN$, over $M \sim \mathcal{L}$, that generalizes to higher dimensions the special complex vector subbundles defined for immersed real surfaces into Kähler surfaces given in [8]. Namely, for each point $p \in M \sim \mathcal{L}$, we define the J -complex vector subspaces of $T_{F(p)}N$

$$\mathcal{F}_p^+ = \{X - JJ_\omega X : X \in T_p M\}, \quad \mathcal{F}_p^- = (\mathcal{F}^+)^{\perp} \quad (4.1)$$

and the linear morphisms, over $M \sim \mathcal{L}$, $\Psi^\pm : TM \rightarrow \mathcal{F}^\pm$, $\Psi^\pm(X) = \frac{1}{2}(X \pm JJ_\omega X)$, being Ψ^+ a complex morphism, while Ψ^- is an anti-complex one, both conformal:

$$\Psi^\pm \circ J_\omega = \pm J \circ \Psi^\pm, \quad g(\Psi^\pm(X), \Psi^\pm(Y)) = \frac{(1 \pm \cos \theta)}{2} g(X, Y) \quad \forall X, Y.$$

In particular Ψ^+ is an isomorphism over $M \sim \mathcal{L}$, what implies \mathcal{F}^+ to be smooth of real rank $2n$. Thus, the same holds for \mathcal{F}^- . Denoting the decompositions $TM^c = T^{1,0}M \oplus T^{0,1}M$ and $TN^c = T^{1,0}N \oplus T^{0,1}N$, with respect to J_ω and J respectively, we have $\Psi^+ : T^{1,0}M \rightarrow T^{1,0}N$, $\Psi^- : T^{1,0}M \rightarrow T^{0,1}N$. At $p \in M \sim \mathcal{L}$, $\forall X \in T_p M$ $X = \Psi^+(X) + \Psi^-(X)$, $JJ_\omega X = \Psi^-(X) - \Psi^+(X)$. Note that, w.r.t the complex structure J , $T_p^{1,0}N = (\mathcal{F}_p^+)^{1,0} \oplus (\mathcal{F}_p^-)^{1,0}$. Then we may take a local unitary o.n. frame $(\sqrt{2}W_\alpha, \sqrt{2}K_\alpha)_{1 \leq \alpha \leq n}$ of $T^{(1,0)}N$, along $M \sim \mathcal{L}$ s.t.

$$W_\alpha \in T^{1,0}N \cap (\mathcal{F}^+)^c \quad K_\alpha \in T^{1,0}N \cap (\mathcal{F}^-)^c. \quad (4.2)$$

Let p a non Lagrangian point, and $X_\alpha, Y_\alpha = J_\omega X_\alpha$ a diagonalising o.n. local frame of $F^*\omega$, on a neighbourhood of p and let $Z_\alpha = \alpha$ as in (2.1). Note that $Z_\alpha \in T^{1,0}M$ with respect to J_ω . Define some local complex maps $u_{\alpha\beta}, v_{\alpha\beta}$ on $M \sim \mathcal{L}$ by

$$\Psi^+(\alpha) = (Z_\alpha)^{1,0} = \sum_\beta u_{\alpha\beta} W_\beta, \quad \Psi^-(\alpha) = (Z_\alpha)^{0,1} = \sum_\beta v_{\alpha\beta} K_{\bar{\beta}}. \quad (4.3)$$

Consider the $n \times n$ complex matrices $u = [u_{\alpha\beta}]_{1 \leq \alpha, \beta \leq n}$, $v = [v_{\alpha\beta}]_{1 \leq \alpha, \beta \leq n}$.

Lemma 4.1. $u \cdot \bar{u}^t = \bar{u}^t \cdot u = \frac{1}{2}(1 + \cos \theta)Id$, $v \cdot \bar{v}^t = \bar{v}^t \cdot v = \frac{1}{2}(1 - \cos \theta)Id$. In particular, for each α, μ , $|u_{\alpha\mu}|^2 \leq \frac{1+\cos \theta}{2}$, $|v_{\alpha\mu}|^2 \leq \frac{1-\cos \theta}{2}$.

Proof. We have $\sum_{\gamma} \frac{1}{2} u_{\alpha\gamma} \overline{u_{\beta\gamma}} = \sum_{\gamma\rho} g(u_{\alpha\gamma} W_{\gamma}, \overline{u_{\beta\rho}} W_{\bar{\rho}}) = g(\Psi^+(\alpha), \overline{\Psi^+(\beta)}) = \frac{1}{2}(1 + \cos \theta) \frac{\delta_{\alpha\beta}}{2}$, and similar for v . Recall that for matrices, $A\bar{A}^t = D$, where D is a real diagonal matrix, implies $\bar{A}^t A = D$. \square

Now we obtain some estimates:

Lemma 4.2. *On a neighbourhood of a point $p \in M \sim \mathcal{L}$, there exists a constant $C > 0$ s.t.*

$$\forall \beta, \mu \text{ and } \forall A, B \in C^\infty(T^c N) \quad |R^N(\beta, \mu, A, B)| \leq C \|\Phi\| \quad (4.4)$$

$$\|\nabla J_\omega\| \leq C \|\Phi\|. \quad (4.5)$$

Proof. Since (N, J, g) is Kähler, by (4.3)

$$R^N(\beta, \mu, A, B) = \sum_{\alpha, \rho} u_{\beta\alpha} v_{\mu\rho} R^N(W_\alpha, K_{\bar{\rho}}, A, B) + v_{\beta\alpha} u_{\mu\rho} R^N(K_\alpha, W_\rho, A, B)$$

Thus, the estimate of $|v_{\mu\rho}|$ in Lemma 4.1 and that $\|\Phi\|^2 = 4 \sin^2 \theta = 4(1 - \cos \theta)(1 + \cos \theta)$ implies (4.4). By (3.3), to estimate $\|\nabla J_\omega\|$ we need only to estimate $|\langle \nabla_Z J_\omega(\mu), \rho \rangle|$. We have

$$\nabla_Z F^* \omega(\mu, \rho) = \langle \nabla_Z (F^* \omega)^\sharp(\mu), \rho \rangle = \langle d \cos \theta(Z) J_\omega(\mu) + \cos \theta \nabla_Z J_\omega(\mu), \rho \rangle = \cos \theta \langle \nabla_Z J_\omega(\mu), \rho \rangle$$

and from (2.6) we obtain (4.5). \square

Proposition 4.1. *Assume F with parallel mean curvature and e.k.as.*

(1) *Locally there exist a constant $C > 0$ such that $\|\Delta \Phi\| \leq C \|\Phi\|$.*

(2) *If N is KE, locally there exist a constant $C > 0$ such that $\|\Delta F^* \omega\| \leq C \|F^* \omega\|$.*

Proof. Note first that by the expression of $\nabla_Z F^* \omega$ in (2.6), we have $\|\nabla F^* \omega\|, \|\delta F^* \omega\| \leq C \|\Phi\|$. The term $\nabla_X^\perp \omega^\perp(H)$ of $\Delta \Phi(X)$ in Prop.3.1(5) can be estimate using (2.7): $\|\nabla_X^\perp \omega^\perp(H)\| \leq C \|\Xi\| \leq C \|\Phi\|$. Now let $L(X) = \sum_i (R^N(e_i, X, (F^* \omega)^\sharp(e_i)) - R^N(e_i, (F^* \omega)^\sharp(X), e_i))^\perp$. L vanish at Lagrangian points. On $M \sim \mathcal{L}$ we take a local unitary complex frame $\sqrt{2}U_\alpha, \sqrt{2}U_{\bar{\alpha}}$ of the complexified normal bundle. Then

$$L(\mu) = \sum_{\alpha} -4i \cos \theta (R^N(\alpha, \mu, \bar{\alpha}))^\perp = \sum_{\alpha, \gamma} -8i \cos \theta R^N(\alpha, \mu, \bar{\alpha}, U_\gamma) U_{\bar{\gamma}} - 8i \cos \theta R^N(\alpha, \mu, \bar{\alpha}, U_{\bar{\gamma}}) U_\gamma.$$

Therefore, by Lemma 4.2 and Lemma 4.1, we conclude that $\|L(\mu)\| \leq C \|\Phi\|$, and we have proved that $\|L(X)\| \leq C \|\Phi\|$. By Prop.3.1(5), away from \mathcal{L} , $\|\Delta \Phi\| \leq C \|\Phi\|$. At a point $p \in \mathcal{L}$, Φ is an isometry and is smooth so the inequality also holds. Again by Prop.3.1(5), $\|\Delta F^* \omega\| \leq C(\|F^* \omega\| + \|\omega^\perp\|) \leq C |\cos \theta| \leq C \|F^* \omega\|$. \square

In order to conclude from Proposition 4.1 that complex points and Lagrangian points are zeros of finite order of Φ and $F^* \omega$ respectively, we need to translate some inequalities of Aronszajn-type for vector bundle maps to similar inequalities for the components. If ψ is a r -form on M with values on a Riemannian vector bundle E over M , the Weitzenböck formula reads

$$\Delta \psi = (d\delta + \delta d)\psi = \sum_i -\nabla_{e_i}(\nabla_{e_i} \psi) + \nabla_{\nabla_{e_i} e_i} \psi_A + S(\psi)$$

where e_i is any o.n. frame of M and $S(\psi)$ is the Weitzenböck operator on $\bigwedge^r T^*M \otimes E$. Assume M is a connected Riemannian manifold, and E_A is a finite family of Riemannian vector bundles over M , and $\forall A, \psi_A \in C^\infty(\bigwedge^{r_A} T^*M \otimes E_A)$ is E_A -valued r_A -form on M . We need the following Aronszajn-type theorem:

Lemma 4.3. *Assume M is connected and there is a constant $C > 0$ s.t. $\|\Delta\psi_A\| \leq \sum_B C(\|\psi_B\| + \|\nabla\psi_B\|) \forall A$. If $\{\psi_A\}$ have a common zero of infinite order, then all $\psi_A \equiv 0$ on all M .*

Proof. Let e_i and $w_{A,\alpha}$ be a local o.n. frames of M and of E_A respectively. For each $\sigma = \{i_1 < \dots < i_{r_A}\}$, $e_{A,\alpha}^\sigma = e_*^{i_1} \wedge \dots \wedge e_*^{i_{r_A}} \otimes w_{A,\alpha}$, defines an o.n. frame of $\bigwedge^{r_A} T^*M \otimes E_A$ for the Hilbert-Schmidt inner product. Let $a_{A,\sigma}^\alpha$ be the local components of ψ_A w.r.t $e_{A,\alpha}^\sigma$, $\psi_A = \sum_{\sigma,\alpha} a_{A,\sigma}^\alpha e_{A,\alpha}^\sigma$. Then $\nabla_X \psi_A = \sum_{\alpha,\sigma} da_{A,\sigma}^\alpha(X) e_{A,\alpha}^\sigma + a_{A,\sigma}^\alpha \nabla_X e_{A,\alpha}^\sigma$ and applying Weitzenböck formula to ψ_A ,

$$\Delta a_{A,\sigma}^\alpha = -\langle \Delta\psi_A, e_{A,\alpha}^\sigma \rangle - \sum_{i\beta\rho} 2da_{A,\rho}^\beta(e_i) \langle \nabla_{e_i} e_{A,\beta}^\rho, e_{A,\alpha}^\sigma \rangle + \sum_{\beta\rho} a_{A,\rho}^\beta \langle \Delta e_{A,\beta}^\rho, e_{A,\alpha}^\sigma \rangle.$$

Consequently, there exists constant $C', C > 0$ s.t. $|\Delta a_{A,\sigma}^\alpha| \leq \|\Delta\psi_A\| + \sum_{\beta,\rho} C'(|a_{A,\rho}^\beta| + \|\nabla a_{A,\rho}^\beta\|) \leq \sum_{B,\beta\rho} C(|a_{B,\rho}^\beta| + \|\nabla a_{B,\rho}^\beta\|)$. A common zero of infinite order of $\{\psi_A\}$ is a common zero of infinite order of the family $\{a_{B,\sigma}^\alpha\}$, and the lemma follows from last remark of [3]. \square

Proposition 4.2. *Let m be the dimension of M and Z be the set of common zeros of ψ_A . If for each $p \in Z$ there exist an A s.t. p is a zero of finite order of ψ_A , then Z is a countably $(m-1)$ -rectifiable set, and in particular has Hausdorff codimension at least 1.*

Proof. See, for example, a proof in [5]. \square

Since the zeros of infinite order of Φ (resp. $F^*\omega$) are zeros of infinite order of $\sin^2 \theta$ (resp. $\cos^2 \theta$), and that we call by complex points (resp. Lagrangian points) of infinite order, the above estimates in Prop.4.1, Lemma 4.3 and Prop.4.2 leads to the conclusion (1) and (2) below:

Corollary 4.1. *Let $F : M^{2n} \rightarrow N^{2n}$ be an immersion with parallel mean curvature and e.k.as. Then:*

- (1) *If $H = 0$, and F is not a complex submanifold, the set \mathcal{C} of complex points is a set of M of Hausdorff codimension at least 1.*
- (2) *If N is KE and F is not a Lagrangian submanifold, the Lagrangian points is a set of M of Hausdorff codimension at least 1.*
- (3) *If $n = 2$, M is closed, and F is any immersion with e.k.as, the set \mathcal{L} of Lagrangian points is a countably $(n-2)$ -rectifiable set and so has Hausdorff codimension at least 2.*

Proof. (3) If $n = 2$ and F has e.k.as then $F^*\omega$ satisfies $DF^*\omega = 0$ where $D = d + \delta$ is the usual Dirac operator for forms on M , from [5] we obtain the result. \square

Remark. In case (3) $F^*\omega$ is an harmonic self-dual 2-form. It is known that the zero set of generic harmonic self-dual 2-form, is a disjoint union of curves diffeomorphic to S^1 .

5 A residue-type formula

5.1 Curvature tensors and characteristic classes

Recall that if (E, g, ∇^E) is a rank k Riemannian vector bundle over a Riemannian manifold M of dimension 4, the first Pontrjagin class $p_1(E)$ can be represented in the cohomology class $H^4(M, \mathbb{R})$ by the 4-form defined by using the curvature tensor R^E of ∇^E

$$p_1(E) = p_1(R^E) = \sum_{i < j} \frac{1}{4\pi^2} R_{ij}^E \wedge R_{ij}^E$$

where the curvature components $R_{ij}^E \in C^\infty(\bigwedge^2 T^*M)$ are defined w.r.t. a local o.n. frame $B = \{E_i\}_{1 \leq i \leq k}$. If $k = 4$ and E is oriented, the Euler characteristic class $\mathcal{X}(E)$ is given by

$$\mathcal{X}(E) = \mathcal{X}(R^E) = \frac{1}{4\pi^2} (R_{12}^E \wedge R_{34}^E - R_{13}^E \wedge R_{24}^E + R_{14}^E \wedge R_{23}^E).$$

If we take $\{1, 2, 3\} = \Lambda_1^\pm, \Lambda_2^\pm, \Lambda_3^\pm$ the usual corresponding basis of $\bigwedge_\pm^2 E$ built from B (see (3.14) for the selfdual case) we easily verify that $R_{12}^{\wedge_\pm^2} = 2R_{\Lambda_3^\pm}^E$, $R_{13}^{\wedge_\pm^2} = -2R_{\Lambda_2^\pm}^E$, $R_{23}^{\wedge_\pm^2} = 2R_{\Lambda_1^\pm}^E$. With this basis one derives the well known relation:

$$p_1(\bigwedge_\pm^2 E) = p_1(E) \pm 2\mathcal{X}(E). \quad (5.1)$$

Set, for direct orthonormal bases e_i of $T_p M$ and E_i of E_p ,

$$z_1 = \frac{1}{2}(e_1 - ie_2), \quad z_2 = \frac{1}{2}(e_3 - ie_4), \quad w_1 = \frac{1}{2}(E_1 - iE_2), \quad w_2 = \frac{1}{2}(E_3 - iE_4)$$

Then $\text{Vol}_M(z_1, z_{\bar{1}}, z_2, z_{\bar{2}}) = -\frac{1}{4}$, and if R_{AB}^E denotes the curvature components w.r.t. this basis, i.e. with $A, B \in \{w_1, w_{\bar{1}}, w_2, w_{\bar{2}}\}$, we have

$$\mathcal{X}(E) = \frac{1}{\pi^2} (R_{12}^E \wedge R_{\bar{1}\bar{2}}^E - R_{1\bar{1}}^E \wedge R_{2\bar{2}}^E + R_{12}^E \wedge R_{2\bar{1}}^E) \quad (5.2)$$

$$p_1(E) = \frac{1}{\pi^2} (-R_{1\bar{1}}^E \wedge R_{1\bar{1}}^E - R_{2\bar{2}}^E \wedge R_{2\bar{2}}^E + 2R_{12}^E \wedge R_{\bar{1}\bar{2}}^E - 2R_{12}^E \wedge R_{2\bar{1}}^E) \quad (5.3)$$

$$p_1(\bigwedge_+^2 E) = -\frac{1}{\pi^2} ((R_{1\bar{1}}^E + R_{2\bar{2}}^E) \wedge (R_{1\bar{1}}^E + R_{2\bar{2}}^E) - 4R_{12}^E \wedge R_{\bar{1}\bar{2}}^E) \quad (5.4)$$

$$p_1(\bigwedge_-^2 E) = -\frac{1}{\pi^2} ((R_{1\bar{1}}^E - R_{2\bar{2}}^E) \wedge (R_{1\bar{1}}^E - R_{2\bar{2}}^E) + 4R_{12}^E \wedge R_{2\bar{1}}^E) \quad (5.5)$$

Herman Weyl introduced some curvature invariants $\kappa_{2c}(M)$, $1 \leq c \leq [\frac{n}{2}]$, of a manifold M of dimension n embedded in a Euclidean space, that appear in its formula on the volume of a tube of radius r about M . These invariants are defined in the same way for any Riemannian manifold M (see e.g. [11]). For $c = 1$, and $c = 2$ they are respectively

$$\kappa_2(M) = \frac{1}{2} \int_M s^M \text{Vol}_M, \quad \kappa_4(M) = \frac{1}{8} \int_M ((s^M)^2 - 4\|Ricci^M\|^2 + \|R^M\|^2) \text{Vol}_M$$

where $\|R^M\|$ is the Hilbert-Schmidt norm of R^M as a 4-tensor. Thus, for $\dim(M) = 4$, κ_4 reads the Gauss-Bonnet formula $\frac{1}{4\pi^2} \kappa_4(M) = \mathcal{X}(M)$ (see e.g [6]).

If E and F are vector bundles over M and $T : TM \rightarrow E$ a 1-tensor, $l : E \times F \rightarrow \mathbb{R}_M$ and $R : TM \times TM \rightarrow F$ 2-tensors, then $l(T \wedge R) \in \bigotimes^3 TM^*$ denotes the 3-tensor

$$l(T \wedge R)(X, Y, Z) = \bigoplus_{X, Y, Z} l(T(X), R(Y, Z)) = l(T(X), R(Y, Z)) + l(T(Z), R(X, Y)) + l(T(Y), R(Z, X)).$$

If R is symmetric (resp. skew symmetric), then so it is $l(T \wedge R)$. We also recall the Kulkarni-Nomizu operator, a symmetric product for two 2-tensors $\phi, \xi \in \bigotimes^2 TM^*$

$$\phi \bullet \xi(X, Y, Z, W) = \phi(X, Z)\xi(Y, W) + \phi(Y, W)\xi(X, Z) - \phi(X, W)\xi(Y, Z) - \phi(Y, Z)\xi(X, W)$$

Assume (E, g_E) is a Riemannian vector bundle with a Riemannian connection ∇^E . The curvature tensor \bar{R} of $TM^* \otimes E$ is given by

$$(\bar{R}(X, Y)\Phi)(Z) = -\nabla_{X,Y}^2 \Phi(Z) + \nabla_{Y,X}^2 \Phi(Z) = R^E(X, Y)(\Phi(Z)) - \Phi(R^M(X, Y)Z)$$

where $\nabla_{X,Y}^2 \Phi = \nabla_X(\nabla_Y \Phi) - \nabla_{\nabla_X Y} \Phi$, for a smooth section Φ of $TM^* \otimes E$.

Lemma 5.1. *Let (E, ∇^E, g_E) be a rank-4 Riemannian vector bundle over M and $\Phi : TM \rightarrow E$ a conformal morphism, with $g_E(\Phi(X), \Phi(Y)) = hg(X, Y)$, and denote by $\Xi = -h\Phi^{-1}$. Then:*

- (1) $g(\Xi(U), Y) = -g_E(U, \Phi(Y))$, and $g(\nabla_X \Xi(U), Y) = -g(\nabla_X \Phi(Y), U)$.
- (2) $g_E(\nabla_X \Phi(Y), \Phi(Z)) + g_E(\Phi(Y), \nabla_X \Phi(Z)) = dh(X)g(Y, Z)$.
- (3) $d^2\Phi(X, Y, Z) = -(\bar{R}(X, Y)\Phi)(Z) - (\bar{R}(Z, X)\Phi)(Y) - (\bar{R}(Y, Z)\Phi)(X) = -R^E \wedge \Phi(X, Y, Z)$.
- (4) $g_E((\bar{R}(X, Y)\Phi)(Z), \Phi(W)) = -g_E((\bar{R}(X, Y)\Phi)(W), \Phi(Z))$.
- (5) $R^E(X, Y, \Phi(Z), \Phi(W)) = hR^M(X, Y, Z, W) + g_E((\bar{R}(X, Y)\Phi)(Z), \Phi(W))$.

Proof. Using $\Phi \circ \Xi = -hId_E$, (1) and (5) are obvious. (2) is obtained from differentiation of $g_E(\Phi(Y), \Phi(Z)) = hg(Y, Z)$. Since ∇^E is a g_E -Riemannian connection, from (5) we derive (4). (3) follows from the definitions $d\Phi(X, Y) = \nabla_X \Phi(Y) - \nabla_Y \Phi(X)$ and $d^2\Phi(X, Y, Z) = \bigoplus_{X, Y, Z} (\nabla_X d\Phi)(Y, Z)$, and that R^M satisfies first Bianchi. \square

We consider the degenerated metric on M , $\hat{g}(X, Y) = g_E(\Phi(X), \Phi(Y))$, and singular connection $\nabla' = \Phi^{-1*}\nabla$ with torsion T that makes $\Phi : (TM, \nabla', \hat{g}) \rightarrow (E, \nabla^E, g_E)$ parallel, namely $\nabla'_X Y = \nabla_X Y + S(X, Y)$, where

$$S(X, Y) = \Phi^{-1}\nabla_X \Phi(Y) \quad \text{and} \quad T(X, Y) = \Phi^{-1}d\Phi(X, Y).$$

It is a Riemannian connection w.r.t \hat{g} . Since Φ is conformal then $\hat{g} = hg$. Let $\hat{\nabla}$ denote the Levi-Civita connection of (M, \hat{g}) , $\varphi = \log h$, and set

$$\hat{S}(X, Y) = \hat{\nabla}_X Y - \nabla_X Y = \frac{1}{2}\varphi_X Y + \frac{1}{2}\varphi_Y X - \frac{1}{2}g(X, Y)\nabla\varphi, \quad S'(X, Y) = \nabla'_X Y - \hat{\nabla}_X Y.$$

where $\varphi_X = d\varphi(X)$. Then $S = \hat{S} + S'$, and $T(X, Y) = S(X, Y) - S(Y, X) = S'(X, Y) - S'(Y, X)$. The curvature tensor $R' : \bigwedge^2 TM \rightarrow \bigwedge^2 TM$ of ∇' , that is given by $\Phi(R^E)$, i.e.

$$\begin{aligned} R'(X, Y, Z, W) &= \hat{g}(R'(X, Y)Z, W) = g_E(R^E(X, Y)\Phi(Z), \Phi(W)) \\ &= hg(\Phi^{-1}R^E(X, Y)\Phi(Z), W) = \hat{g}(\Phi(R^E)(X, Y)Z, W) \end{aligned} \quad (5.6)$$

may not be a curvature-type tensor. The Bianchi map for $R : \bigwedge^2 TM \rightarrow \bigwedge^2 TM$ is defined as $g(b(R)(X, Y, Z), W) = \bigoplus_{X, Y, Z} R(X, Y, Z, W)$. Note that $b(R) \in \bigwedge^3 TM^* \otimes TM \cap L(\bigwedge^2 TM; \bigwedge^2 TM)$.

Proposition 5.1. *In the conditions of previous lemma, $b(R') = -\Phi^{-1}d^2\Phi$. So R' satisfies the first Bianchi identity iff $(\bar{R}(X, Y)\Phi)(Z)$ does so, iff $d^2\Phi = 0$. In that case R' is also symmetric. Thus, $R' = \Phi(R^E)$ is a curvature operator at a point $p \in M$ (i.e $R' \in \mathcal{B}$, see notation in [25]) iff $d^2\Phi(p) = 0$.*

Proof. Using the fact that R^M satisfies the first Bianchi identity and lemma 5.1(3) gives

$$\begin{aligned} b(R')(X, Y, Z, W) &= \bigoplus_{X, Y, Z} R^E(X, Y, \Phi(Z), \Phi(W)) = \bigoplus_{X, Y, Z} g_E((\bar{R}(X, Y)\Phi)(Z), \Phi(W)) \\ &= -g_E(d^2\Phi(X, Y, Z), \Phi(W)) = -\hat{g}(\Phi^{-1}d^2\Phi(X, Y, Z), W). \end{aligned}$$

Now we have from symmetry of R^M ,

$$\begin{aligned} R'(Z, W, X, Y) - R'(X, Y, Z, W) &= g((\bar{R}(X, Y)\Phi)(Z), \Phi(W)) - g((\bar{R}(Z, W)\Phi)(X), \Phi(Y)) \\ &= -g_E(d^2\Phi(X, Y, Z), \Phi(W)) + g_E(d^2\Phi(X, Y, W), \Phi(Z)) \\ &\quad - g_E(d^2\Phi(Y, Z, W), \Phi(X)) + g_E(d^2\Phi(X, Z, W), \Phi(Y)). \quad \square \end{aligned}$$

Proposition 5.2. *If $(Y, Z) \rightarrow g(\nabla_X\Phi(Y), \Phi(Z))$ is symmetric then $\bar{R}(X, Y)\Phi = 0$.*

Proof. Set $\varphi = \log h$. From lemma 5.1(2) we have $\nabla_X\Phi(Y) = \frac{1}{2}d\varphi(X)\Phi(Y)$. It follows that $\nabla_{X,Y}^2\Phi(Z) = \frac{1}{2}Hess\varphi(X, Y)\Phi(Z) + \frac{1}{4}\varphi_X\varphi_Y\Phi(Z)$. That implies $\nabla_{X,Y}^2\Phi = \nabla_{Y,X}^2\Phi$. \square

5.2 Proof of (1.5) of Theorem 1.1

If N is KE with $Ricci^N = Rg$, (3.4) says that

$$\sum_{\alpha} R^N(\alpha, \Phi(\bar{\alpha})) = \frac{\sin^2 \theta}{4} R\omega = \sum_{\alpha} R^N(\bar{\alpha}, \Phi(\alpha)) \quad (5.7)$$

Note that $JX = (JX)^{\top} + (JX)^{\perp} = \cos \theta J_{\omega}(X) + \Phi(X)$, and so

$$\begin{aligned} R^N(X, Y, \Phi(Z), \Phi(W)) &= R^N(X, Y, Z, W) + \cos^2 \theta R^N(X, Y, J_{\omega}Z, J_{\omega}W) \\ &\quad + \cos \theta R^N(X, Y, Z, JJ_{\omega}W) + \cos \theta R^N(X, Y, JJ_{\omega}Z, W) \end{aligned}$$

Since $J_{\omega}(\alpha) = i\alpha$, we have on $M \sim \mathcal{L}$,

$$\begin{aligned} R^N(\Phi(\alpha), \Phi(\beta)) &= \sin^2 \theta R^N(\alpha, \beta), \quad R^N(\Phi(\alpha), \bar{\gamma}) = -R^N(\alpha, \Phi(\bar{\gamma})), \\ R^N(\Phi(\alpha), \gamma) &= -R^N(\alpha, \Phi(\gamma)) - 2i \cos \theta R^N(\alpha, \gamma), \\ R^N(\Phi(\alpha), \Phi(\bar{\gamma})) &= (1 + \cos^2 \theta) R^N(\alpha, \bar{\gamma}) - 2i \cos \theta R^N(\alpha, J\bar{\gamma}) = \sin^2 \theta R^N(\alpha, \bar{\gamma}) - 2i \cos \theta R^N(\alpha, \Phi(\bar{\gamma})). \end{aligned}$$

$$\sum_{\alpha} R^N(\Phi(\alpha), \Phi(\bar{\alpha})) = \sum_{\alpha} \sin^2 \theta R^N(\alpha, \bar{\alpha}) - 2i \cos \theta R^N(\alpha, \Phi(\bar{\alpha})) = \sin^2 \theta \left(\sum_{\alpha} R^N(\alpha, \bar{\alpha}) - \frac{i}{2} \cos \theta R\omega \right). \quad (5.8)$$

The Gauss and the Ricci equations (2.8)-(2.9) gives

Lemma 5.2. *On $M \sim \mathcal{L}$*

$$R^\perp(\Phi(\alpha), \Phi(\beta)) = \sin^2 \theta R^M(\alpha, \beta) - \sin^2 \theta \langle \nabla_\alpha dF \wedge \nabla_\beta dF \rangle + \langle A^{\Phi(\alpha)} \wedge A^{\Phi(\beta)} \rangle \quad (5.9)$$

$$\begin{aligned} R^\perp(\Phi(\alpha), \Phi(\bar{\beta})) &= \sin^2 \theta R^M(\alpha, \bar{\beta}) - 2i \cos \theta R^N(\alpha, \Phi(\bar{\beta})) \\ &\quad - \sin^2 \theta \langle \nabla_\alpha dF \wedge \nabla_{\bar{\beta}} dF \rangle + \langle A^{\Phi(\alpha)} \wedge A^{\Phi(\bar{\beta})} \rangle. \end{aligned} \quad (5.10)$$

We have for $A, B \in T_p^c M$

$$\begin{aligned} &\langle \nabla_A dF \wedge \nabla_B dF \rangle(X, Y) - \frac{1}{\sin^2 \theta} \langle A^{\Phi(A)} \wedge A^{\Phi(B)} \rangle(X, Y) = \\ &= \frac{2}{\sin^2 \theta} \sum_{\alpha} \left((g_X A \alpha g_Y B \bar{\alpha} - g_X \alpha A g_Y \bar{\alpha} B) + (g_X A \bar{\alpha} g_Y B \alpha - g_X \bar{\alpha} A g_Y \alpha B) \right. \\ &\quad \left. - (g_Y A \alpha g_X B \bar{\alpha} - g_Y \alpha A g_X \bar{\alpha} B) - (g_Y A \bar{\alpha} g_X B \alpha - g_Y \bar{\alpha} A g_X \alpha B) \right) \end{aligned}$$

Using lemma 3.1 applied to the above equation we have

Lemma 5.3. *For a Cayley submanifold $F : M \rightarrow N$, we have on $M \sim (\mathcal{L} \cup \mathcal{C})$,*

$$\begin{aligned} \frac{1}{2} \langle A^{\Phi(1)} \wedge A^{\Phi(\bar{1})} \rangle &= \frac{\sin^2 \theta}{2} \langle \nabla_1 dF \wedge \nabla_{\bar{1}} dF \rangle + \frac{i}{2} d \cos \theta \wedge (g \cdot \bar{1}1 + g \cdot 1\bar{1}) \\ &\quad + \cos \theta g \cdot 21 \wedge g(\nabla \cdot J_\omega(\bar{1}), \bar{2}) - \cos \theta g \cdot \bar{1}2 \wedge g(\nabla \cdot J_\omega(1), 2) \\ \frac{1}{2} \langle A^{\Phi(2)} \wedge A^{\Phi(\bar{2})} \rangle &= \frac{\sin^2 \theta}{2} \langle \nabla_2 dF \wedge \nabla_{\bar{2}} dF \rangle + \frac{i}{2} d \cos \theta \wedge (g \cdot \bar{2}2 + g \cdot 2\bar{2}) \\ &\quad + \cos \theta g \cdot 12 \wedge g(\nabla \cdot J_\omega(\bar{2}), \bar{1}) - \cos \theta g \cdot \bar{2}1 \wedge g(\nabla \cdot J_\omega(2), 1) \\ \frac{1}{2} \langle A^{\Phi(1)} \wedge A^{\Phi(2)} \rangle &= \frac{\sin^2 \theta}{2} \langle \nabla_1 dF \wedge \nabla_2 dF \rangle \\ \frac{1}{2} \langle A^{\Phi(1)} \wedge A^{\Phi(\bar{2})} \rangle &= \frac{\sin^2 \theta}{2} \langle \nabla_2 dF \wedge \nabla_{\bar{2}} dF \rangle + i d \cos \theta \wedge g \cdot 1\bar{2} \\ &\quad - \cos \theta g \cdot 11 \wedge g(\nabla \cdot J_\omega(\bar{1}), \bar{2}) - \cos \theta g \cdot \bar{2}2 \wedge g(\nabla \cdot J_\omega(1), 2) \end{aligned}$$

Proof. This is a long but straightforward proof using lemma 3.1(5)(6)(8) and (3.3). We only prove one of the equalities, for the other ones are similar.

$$\begin{aligned} &\left(\frac{\sin^2 \theta}{2} \langle \nabla_1 dF \wedge \nabla_2 dF \rangle - \frac{1}{2} \langle A^{\Phi(1)} \wedge A^{\Phi(2)} \rangle \right)(X, Y) = \\ &= \sum_{\alpha} (g_X 1 \alpha g_Y 2 \bar{\alpha} - g_X \alpha 1 g_Y \bar{\alpha} 2) + (g_X 1 \bar{\alpha} g_Y 2 \alpha - g_X \bar{\alpha} 1 g_Y \alpha 2) \\ &\quad + \sum_{\alpha} (-g_Y 1 \alpha g_X 2 \bar{\alpha} + g_Y \alpha 1 g_X \bar{\alpha} 2) + (-g_Y 1 \bar{\alpha} g_X 2 \alpha + g_Y \bar{\alpha} 1 g_X \alpha 2) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha} g_X 1 \alpha g_Y 2 \bar{\alpha} - (g_X 1 \alpha + \cos \theta g(\nabla_X J_{\omega}(1), \alpha))(g_Y 2 \bar{\alpha} + \frac{i}{2} \delta_{\alpha 2} d \cos \theta(Y)) \\
&\quad + g_X 1 \bar{\alpha} g_Y 2 \alpha - (g_X 1 \bar{\alpha} + \frac{i}{2} \delta_{\alpha 1} d \cos \theta(X))(g_Y 2 \alpha + \cos \theta g(\nabla_Y J_{\omega}(2), \alpha)) \\
&\quad \sum_{\alpha} -g_Y 1 \alpha g_X 2 \bar{\alpha} + (g_Y 1 \alpha + \cos \theta g(\nabla_Y J_{\omega}(1), \alpha))(g_X 2 \bar{\alpha} + \frac{i}{2} \delta_{\alpha 2} d \cos \theta(X)) \\
&\quad - g_Y 1 \bar{\alpha} g_X 2 \alpha + (g_Y 1 \bar{\alpha} + \frac{i}{2} \delta_{\alpha 1} d \cos \theta(Y))(g_X 2 \alpha + \cos \theta g(\nabla_X J_{\omega}(2), \alpha)) \\
&= -\frac{i}{2} g_X 1 2 d \cos \theta(Y) - \cos \theta g(\nabla_X J_{\omega}(1), 2) g_Y 2 \bar{2} - \cos \theta g_X 1 \bar{1} g(\nabla_Y J_{\omega}(2), 1) - \frac{i}{2} d \cos \theta(X) g_Y 2 1 \\
&\quad + \frac{i}{2} g_Y 1 2 d \cos \theta(X) + \cos \theta g(\nabla_Y J_{\omega}(1), 2) g_X 2 \bar{2} + \cos \theta g_Y 1 \bar{1} g(\nabla_X J_{\omega}(2), 1) + \frac{i}{2} d \cos \theta(Y) g_X 2 1 \\
&\quad - \frac{i}{2} \cos \theta d \cos \theta(Y) g(\nabla_X J_{\omega}(1), 2) - \frac{i}{2} \cos \theta d \cos \theta(X) g(\nabla_Y J_{\omega}(2), 1) \\
&\quad + \frac{i}{2} \cos \theta d \cos \theta(X) g(\nabla_Y J_{\omega}(1), 2) + \frac{i}{2} \cos \theta d \cos \theta(Y) g(\nabla_X J_{\omega}(2), 1) \\
&= \frac{i}{2} d \cos \theta(X) (-g_Y 2 1 + g_Y 1 2) - \frac{i}{2} d \cos \theta(Y) (-g_X 2 1 + g_X 1 2) \\
&\quad - \cos \theta g(\nabla_X J_{\omega}(1), 2) g_Y 2 \bar{2} - \cos \theta g(\nabla_Y J_{\omega}(2), 1) g_X 1 \bar{1} \\
&\quad + \cos \theta g(\nabla_Y J_{\omega}(1), 2) g_X 2 \bar{2} + \cos \theta g(\nabla_X J_{\omega}(2), 1) g_Y 1 \bar{1} \\
&\quad + i \cos \theta d \cos \theta(X) g(\nabla_Y J_{\omega}(1), 2) - i \cos \theta d \cos \theta(Y) g(\nabla_X J_{\omega}(1), 2) \\
&= \frac{i}{2} \cos \theta d \cos \theta(X) g(\nabla_Y J_{\omega}(2), 1) - \frac{i}{2} \cos \theta d \cos \theta(Y) g(\nabla_X J_{\omega}(2), 1) \\
&\quad - \cos \theta g(\nabla_X J_{\omega}(1), 2) (g_Y 2 \bar{2} + g_Y 1 \bar{1}) + \cos \theta g(\nabla_Y J_{\omega}(1), 2) (g_X 2 \bar{2} + g_X 1 \bar{1}) \\
&\quad + i \cos \theta d \cos \theta(X) g(\nabla_Y J_{\omega}(1), 2) - i \cos \theta d \cos \theta(Y) g(\nabla_X J_{\omega}(1), 2) \\
&= -\frac{i}{2} \cos \theta d \cos \theta(Y) g(\nabla_X J_{\omega}(1), 2) + \frac{i}{2} \cos \theta d \cos \theta(X) g(\nabla_Y J_{\omega}(1), 2) \\
&\quad + \cos \theta g(\nabla_X J_{\omega}(1), 2) (\frac{i}{2} d \cos \theta(Y)) + \cos \theta g(\nabla_Y J_{\omega}(1), 2) (-\frac{i}{2} d \cos \theta(X)) \\
&= 0 \quad \square.
\end{aligned}$$

The two previous lemmas, (5.7) and (5.8) give us

Proposition 5.3. *If $F : M \rightarrow N$ is a Cayley submanifold of a Ricci-flat N , then on $M \sim \mathcal{L}$*

$$\begin{aligned}
R^{\perp}(\Phi(1), \Phi(\bar{1})) &= \sin^2 \theta R^M(1, \bar{1}) + 2id \cos \theta \wedge g \cdot 1 \bar{1} - 2i \cos \theta R^N(1, \Phi(\bar{1})) \\
&\quad + 2 \cos \theta g \cdot 2 1 \wedge g(\nabla \cdot J_{\omega}(\bar{1}), \bar{2}) - 2 \cos \theta g \cdot 1 \bar{2} \wedge g(\nabla \cdot J_{\omega}(1), 2) \\
R^{\perp}(\Phi(2), \Phi(\bar{2})) &= \sin^2 \theta R^M(2, \bar{2}) - 2id \cos \theta \wedge g \cdot 1 \bar{1} - 2i \cos \theta R^N(2, \Phi(\bar{2})) \\
&\quad + 2 \cos \theta g \cdot 1 2 \wedge g(\nabla \cdot J_{\omega}(\bar{2}), \bar{1}) - 2 \cos \theta g \cdot \bar{2} \bar{1} \wedge g(\nabla \cdot J_{\omega}(2), 1) \\
R^{\perp}(\Phi(1), \Phi(2)) &= \sin^2 \theta R^M(1, 2) \\
R^{\perp}(\Phi(1), \Phi(\bar{2})) &= \sin^2 \theta R^M(1, \bar{2}) + 2id \cos \theta \wedge g \cdot 1 \bar{2} - 2i \cos \theta R^N(1, \Phi(\bar{2})) \\
&\quad - 2 \cos \theta g \cdot 1 1 \wedge g(\nabla \cdot J_{\omega}(\bar{1}), \bar{2}) - 2 \cos \theta g \cdot \bar{2} \bar{2} \wedge g(\nabla \cdot J_{\omega}(1), 2)
\end{aligned}$$

Furthermore, $\sum_{\alpha} R^{\perp}(\Phi(\alpha), \Phi(\bar{\alpha})) = \sum_{\alpha} \sin^2 \theta R^M(\alpha, \bar{\alpha})$.

Proposition 5.4. *If $F : M \rightarrow N$ is a non- J -holomorphic Cayley submanifold and N is Ricci-flat, then (1.5) holds, that is $p_1(\bigwedge_+^2 NM) = p_1(\bigwedge_+^2 TM)$.*

Proof. If M is a Lagrangian submanifold, then $\Phi : TM \rightarrow NM$ is an orientation preserving isometry, and so characteristic classes of M and NM are the same. Now we assume M is neither Lagrangian nor complex submanifold. We consider the formulas (5.2)-(5.4) using the curvature tensors R^M of M and R^\perp of NM w.r.t the connections ∇^M of M and ∇^\perp of NM , respectively, and away from complex and Lagrangian points we may take e_1, e_2, e_3, e_4 as X_1, Y_1, X_2, Y_2 and E_1, E_2, E_3, E_4 as $\frac{\Phi(X_1)}{\sin \theta}, \frac{\Phi(Y_1)}{\sin \theta}, \frac{\Phi(X_2)}{\sin \theta}, \frac{\Phi(Y_2)}{\sin \theta}$. By the previous Proposition 5.3 and (5.4) we easily see that the equality (1.5) is valid on $M \sim \mathcal{L} \cup \mathcal{C}$, as forms (and not only as chomology classes). Moreover the expressions in (5.2)-(5.5) do not depend on the o.n. basis used, and are smoothly defined on all M . Since the set of complex and Lagrangian points have empty interior (corollary 4.1). Then (5.4) and so (1.5) stays valid on all M . \square .

From (5.1) and the previous proposition we obtain:

Corollary 5.1. *In the conditions of the Prop.5.4, $\mathcal{X}(M) - \mathcal{X}(NM) = \frac{1}{2}(p_1(NM) - p_1(M))$.*

5.3 Proof. of (1.6) of Theorem 1.1

Since ∇' is a \hat{g} -Riemannian connection, and $\nabla' = \hat{\nabla} + S'$, by Theorem 1.1 of [25], we have

$$p_1(R') - p_1(R^M) = -\frac{1}{2\pi^2} d \left(\langle S' \wedge (\hat{R}^M - \frac{1}{2}dS' - \frac{1}{3}(S')^2) \rangle_{\hat{g}} \right) \quad (5.11)$$

where $S' : TM \rightarrow \wedge^2 TM$, $(S')^2, \hat{R}^M : \wedge^2 TM \rightarrow \wedge^2 TM$ are defined by

$$\begin{aligned} \langle S'(X), Y \wedge Z \rangle_{\hat{g}} &= \hat{g}(S'(X, Y), Z) \\ \langle (S')^2(X \wedge Y), Z \wedge W \rangle_{\hat{g}} &= \hat{g}(S'(X, Z), S'(Y, W)) - \hat{g}(S'(X, W), S'(Y, Z)) \\ \langle \hat{R}^M(X \wedge Y), Z \wedge W \rangle_{\hat{g}} &= h(R^M(X, Y, Z, W) + \phi \bullet g(X, Y, Z, W)) \end{aligned}$$

where \hat{R}^M is the curvature tensor of $(M, \hat{g} = hg)$, and

$$\phi = \frac{1}{2} \left(-\frac{\|\nabla \log h\|^2}{4} g + \frac{1}{2} d \log h \otimes d \log h - Hess(\log h) \right). \quad (5.12)$$

The inner product $\langle \cdot, \cdot \rangle_{\hat{g}}$ is the usual inner product on $\wedge^2 TM$, defined w.r.t. \hat{g} . So we will compute all the terms in (5.11).

Let $\varphi = \log h$. The letters A, B, \dots denote vector fields $X, Y, Z, \nabla \varphi$ or e_A, \dots , and we denote by

$$\Phi_A B = \Phi^{-1} \nabla_A \Phi(B) \quad \Phi_A B C = g(\Phi^{-1} \nabla_A \Phi(B), C) \quad \varphi_A = d\varphi(A).$$

The gradient $\nabla \varphi$ is w.r.t. g . From lemma 5.1,

$$\Phi_A B C + \Phi_A C B = \varphi_A g(B, C). \quad (5.13)$$

We easily derive

$$\varphi_Z \Phi_Y X A - \varphi_Y \Phi_Z X A = -\varphi_Z \Phi_Y A X + \varphi_Y \Phi_Z A X \quad (5.14)$$

$$\Phi_Y A Z - \Phi_Z A Y = g(\Phi^{-1} d\Phi(Z, Y), A) + g(\varphi_Y Z - \varphi_Z Y, A) \quad (5.15)$$

$$\Phi_Y Z \nabla \varphi - \Phi_Z Y \nabla \varphi = -\Phi_Y \nabla \varphi Z + \Phi_Z \nabla \varphi Y = g(\Phi^{-1} d\Phi(Y, Z), \nabla \varphi). \quad (5.16)$$

Now we have

$$\begin{aligned} S'(X, Y) &= \Phi^{-1} \nabla_X \Phi(Y) - \frac{1}{2} \varphi_X Y - \frac{1}{2} \varphi_Y X + \frac{1}{2} g(X, Y) \nabla \varphi \\ \hat{\nabla}_X Y &= \nabla_X Y + \frac{1}{2} \varphi_X Y + \frac{1}{2} \varphi_Y X - \frac{1}{2} g(X, Y) \nabla \varphi. \end{aligned}$$

Since ∇' is a \hat{g} -Riemannian connection and $S'(X, Y) = \nabla'_X Y - \hat{\nabla}_X Y$, then $\hat{g}(S'(X, Y), Z) = -\hat{g}(S'(X, Z), Y)$, and so, the same holds w.r.t. g . Let e_i be a g -o.n. basis of $T_p M$. We have

$$g(S'(X, e_i), e_j) = \Phi_X i j - \frac{1}{2} \varphi_X \delta_{ij} - \frac{1}{2} \varphi_i g(X, e_j) + \frac{1}{2} g(X, e_i) \varphi_j. \quad (5.17)$$

We consider from now on TM with the metric g , and from a tensor $\varrho \in C^\infty(\wedge^2 TM \otimes \wedge^2 TM)$ we define a 4-tensor on M as $\varrho(X, Y, Z, W) = \langle \varrho(X \wedge Y), Z \wedge W \rangle_g$, where $\langle X \wedge Y, Z \wedge W \rangle_g = g(X, Z)g(Y, W) - g(X, W)g(Y, Z)$ is the Riemannian structure in $\wedge^2 TM$ defined w.r.t. g . For each tangent vector X of $T_p M$ we denote by $\hat{X} = h^{-\frac{1}{2}} X$.

Lemma 5.4. *If $\varrho \in C^\infty(\wedge^2 TM \otimes \wedge^2 TM)$ and e_i is a g -o.n. basis of $T_p M$, $X, Y, Z \in T_p M$*

$$\bigoplus_{X, Y, Z} \sum_{ij} g(S'(X, e_i), e_j) \varrho(Y, Z, e_i, e_j) = b(\varrho)(X, Y, Z, \nabla \varphi) - \bigoplus_{X, Y, Z} \sum_i \varrho(Y, Z, \Phi^{-1} \nabla_X \Phi(e_i), e_i).$$

Proof. Since $\varrho(Y, Z, e_i, e_j)$ is skew symmetric on (e_i, e_j)

$$\bigoplus_{X, Y, Z} \sum_{ij} g(S'(X, e_i), e_j) \varrho(Y, Z, e_i, e_j) = \bigoplus_{X, Y, Z} \sum_i \varrho(Y, Z, e_i, \Phi^{-1} \nabla_X \Phi(e_i)) + \varrho(Y, Z, X, \nabla \varphi). \quad \square$$

From previous lemma and (5.13) we obtain for any 2-tensor $\xi \in C^\infty(\otimes^2 TM^*)$

$$\begin{aligned} &\bigoplus_{X, Y, Z} \sum_{ij} g(S'(X, e_i), e_j) \xi \bullet g(Y, Z, e_i, e_j) = \\ &= \bigoplus_{X, Y, Z} -\xi(Y, \Phi^{-1} \nabla_X \Phi(Z)) + \xi(Z, \Phi^{-1} \nabla_X \Phi(Y)) + \sum_i \xi(Y, e_i) \Phi_X i Z - \xi(Z, e_i) \Phi_X i Y \\ &= \bigoplus_{X, Y, Z} -\xi(Y, \Phi^{-1} \nabla_X \Phi(Z)) + \xi(Z, \Phi^{-1} \nabla_X \Phi(Y)) \\ &\quad + \bigoplus_{X, Y, Z} \sum_i \xi(Y, e_i) (-\Phi_X Z i + \varphi_X g(Z, e_i)) - \xi(Z, e_i) (-\Phi_X Y i + \varphi_X g(Y, e_i)) \\ &= \bigoplus_{X, Y, Z} 2\xi(Z, \Phi^{-1} d\Phi(X, Y)). \end{aligned}$$

Thus

Lemma 5.5.

$$\bigoplus_{X, Y, Z} \sum_{ij} g(S'(X, e_i), e_j) R^M(Y, Z, e_i, e_j) = \bigoplus_{X, Y, Z} \sum_i -R^M(Y, Z, \Phi^{-1} \nabla_X \Phi(e_i), e_i) \quad (5.18)$$

$$\bigoplus_{X, Y, Z} \sum_{ij} g(S'(X, e_i), e_j) \xi \bullet g(Y, Z, e_i, e_j) = \bigoplus_{X, Y, Z} 2\xi(X, \Phi^{-1} d\Phi(Y, Z)). \quad (5.19)$$

Lemma 5.6. *Let $\varrho(Y, Z, A, B) = g(S'(Y, A), S'(Z, B)) - g(S'(Y, B), S'(Z, A))$. We have*

$$\begin{aligned} &\bigoplus_{X, Y, Z} \sum_{ij} g(S'(X, e_i), e_j) \varrho(Y, Z, e_i, e_j) = \\ &= \bigoplus_{X, Y, Z} \left(\sum_{ij} (\Phi_X i j - \Phi_X j i) g(\Phi^{-1} \nabla_Y \Phi(e_i), \Phi^{-1} \nabla_Z \Phi(e_j)) - 3g(\Phi^{-1} \nabla_X \Phi(\nabla \varphi), \Phi^{-1} d\Phi(Y, Z)) \right) \\ &\quad + \bigoplus_{X, Y, Z} \left(\frac{3}{2} d\varphi \otimes d\varphi(X, \Phi^{-1} d\Phi(Y, Z)) + \frac{3}{4} \|\nabla \varphi\|^2 g(X, \Phi^{-1} d\Phi(Y, Z)) \right). \end{aligned} \quad (5.20)$$

Proof.

$$g(S'(Y, e_i), S'(Z, e_j)) = g\left(\Phi^{-1}\nabla_Y\Phi(e_i) - \frac{1}{2}\varphi_Y e_i - \frac{1}{2}\varphi_i Y + \frac{1}{2}g(Y, e_i)\nabla\varphi, \right. \\ \left. \Phi^{-1}\nabla_Z\Phi(e_j) - \frac{1}{2}\varphi_Z e_j - \frac{1}{2}\varphi_j Z + \frac{1}{2}g(Z, e_j)\nabla\varphi\right).$$

Using (5.13) and the fact that $g(S'(X, e_i), e_j)$ is skew symmetric on (i, j) we have, after interchanging i with j in some terms,

$$\begin{aligned} & \bigoplus_{x, Y, Z} \sum_{ij} g(S'(X, e_i), e_j) \varrho(Y, Z, e_i, e_j) = \\ & = \bigoplus_{x, Y, Z} \sum_{ij} g(S'(X, e_i), e_j) \left(2g(\Phi^{-1}\nabla_Y\Phi(e_i), \Phi^{-1}\nabla_Z\Phi(e_j)) - \varphi_Z \Phi_Y i j + \varphi_Y \Phi_Z i j \right. \\ & \quad \left. - \varphi_j \Phi_Y i Z + \varphi_j \Phi_Z i Y + g(Z, e_j) \Phi_Y i \nabla\varphi - g(Y, e_j) \Phi_Z i \nabla\varphi \right) \end{aligned} \quad (5.21)$$

$$+ \bigoplus_{x, Y, Z} \sum_{ij} g(S'(X, e_i), e_j) \left(-\frac{3}{4}d\varphi \otimes d\varphi + \frac{1}{8}\|\nabla\varphi\|^2 g \right) \bullet g(Y, Z, e_i, e_j) \quad (5.22)$$

$$(5.21) + (5.22) =$$

$$\begin{aligned} & = \bigoplus_{x, Y, Z} \sum_{ij} \left(\Phi_X i j - \frac{1}{2}\varphi_X \delta_{ij} - \frac{1}{2}\varphi_i g(X, e_j) + \frac{1}{2}g(X, e_i) \varphi_j \right) \left(2g(\Phi^{-1}\nabla_Y\Phi(e_i), \Phi^{-1}\nabla_Z\Phi(e_j)) \right. \\ & \quad \left. - \varphi_Z \Phi_Y i j + \varphi_Y \Phi_Z i j - \varphi_j \Phi_Y i Z + \varphi_j \Phi_Z i Y + g(Z, e_j) \Phi_Y i \nabla\varphi - g(Y, e_j) \Phi_Z i \nabla\varphi \right) \\ & = \bigoplus_{x, Y, Z} \sum_{ij} + 2\Phi_X i j g(\Phi^{-1}\nabla_Y\Phi(e_i), \Phi^{-1}\nabla_Z\Phi(e_j)) - \varphi_Z \Phi_X i j \Phi_Y i j + \varphi_Y \Phi_X i j \Phi_Z i j \\ & \quad + \bigoplus_{x, Y, Z} \sum_i -\Phi_X i \nabla\varphi \Phi_Y i Z + \Phi_X i \nabla\varphi \Phi_Z i Y + \Phi_X i Z \Phi_Y i \nabla\varphi - \Phi_X i Y \Phi_Z i \nabla\varphi \\ & \quad + \bigoplus_{x, Y, Z} \sum_i -\varphi_X g(\Phi^{-1}\nabla_Y\Phi(e_i), \Phi^{-1}\nabla_Z\Phi(e_i)) + \varphi_X \varphi_Z \varphi_Y - \varphi_X \varphi_Y \varphi_Z \\ & \quad + \bigoplus_{x, Y, Z} + \frac{1}{2}\varphi_X \Phi_Y \nabla\varphi Z - \frac{1}{2}\varphi_X \Phi_Z \nabla\varphi Y - \frac{1}{2}\varphi_X \Phi_Y Z \nabla\varphi + \frac{1}{2}\varphi_X \Phi_Z Y \nabla\varphi \end{aligned} \quad (5.24)$$

$$+ \bigoplus_{x, Y, Z} -g(\Phi^{-1}\nabla_Y\Phi(\nabla\varphi), \Phi^{-1}\nabla_Z\Phi(X)) + \frac{1}{2}\varphi_Z \Phi_Y \nabla\varphi X - \frac{1}{2}\varphi_Y \Phi_Z \nabla\varphi X \quad (5.25)$$

$$+ \bigoplus_{x, Y, Z} + \frac{1}{2}\varphi_X \Phi_Y \nabla\varphi Z - \frac{1}{2}\varphi_X \Phi_Z \nabla\varphi Y - \frac{1}{4}g(X, Z) \varphi_Y \|\nabla\varphi\|^2 + \frac{1}{4}g(X, Y) \varphi_Z \|\nabla\varphi\|^2 \quad (5.26)$$

$$+ \bigoplus_{x, Y, Z} + g(\Phi^{-1}\nabla_Y\Phi(X), \Phi^{-1}\nabla_Z\Phi(\nabla\varphi)) - \frac{1}{2}\varphi_Z \Phi_Y X \nabla\varphi + \frac{1}{2}\varphi_Y \Phi_Z X \nabla\varphi \quad (5.27)$$

$$+ \bigoplus_{x, Y, Z} -\frac{1}{2}\|\nabla\varphi\|^2 \Phi_Y X Z + \frac{1}{2}\|\nabla\varphi\|^2 \Phi_Z X Y + \frac{1}{2}\varphi_Z \Phi_Y X \nabla\varphi - \frac{1}{2}\varphi_Y \Phi_Z X \nabla\varphi \quad (5.28)$$

The last two terms of (5.27) cancel with the last two of (5.28). From (5.14), (5.15) and (5.16)

$$\bigoplus_{x, Y, Z} \frac{1}{2}\varphi_Z \Phi_Y \nabla\varphi X - \frac{1}{2}\varphi_Y \Phi_Z \nabla\varphi X = \bigoplus_{x, Y, Z} \frac{1}{2}\varphi_X \Phi_Z \nabla\varphi Y - \frac{1}{2}\varphi_X \Phi_Y \nabla\varphi Z = \bigoplus_{x, Y, Z} \frac{1}{2}\varphi_X g(\Phi^{-1}d\Phi(Y, Z), \nabla\varphi)$$

that we replace in (5.25), and $\frac{1}{2}\varphi_X \Phi_Y \nabla\varphi Z - \frac{1}{2}\varphi_X \Phi_Z \nabla\varphi Y = -\frac{1}{2}\varphi_X \Phi_Y Z \nabla\varphi + \frac{1}{2}\varphi_X \Phi_Z Y \nabla\varphi = \frac{1}{2}\varphi_X g(\Phi^{-1}d\Phi(Z, Y), \nabla\varphi)$ that we replace in (5.24) and (5.26). We also have $\forall ij$

$$\bigoplus_{x, Y, Z} \varphi_Z \Phi_X i j \Phi_Y i j - \varphi_Y \Phi_X i j \Phi_Z i j = 0. \quad (5.29)$$

Thus,

$$(5.21) + (5.22) = \bigoplus_{X,Y,Z} \sum_{ij} + 2\Phi_X ij g(\Phi^{-1}\nabla_Y \Phi(e_i), \Phi^{-1}\nabla_Z \Phi(e_j)) \quad (5.30)$$

$$+ \bigoplus_{X,Y,Z} \sum_i -\Phi_X i \nabla \varphi \Phi_Y i Z + \Phi_X i \nabla \varphi \Phi_Z i Y + \Phi_X i Z \Phi_Y i \nabla \varphi - \Phi_X i Y \Phi_Z i \nabla \varphi \quad (5.31)$$

$$+ \bigoplus_{X,Y,Z} \sum_i -\varphi_X g(\Phi^{-1}\nabla_Y \Phi(e_i), \Phi^{-1}\nabla_Z \Phi(e_i)) \quad (5.32)$$

$$+ \bigoplus_{X,Y,Z} -\varphi_X g(\Phi^{-1}d\Phi(Y, Z), \nabla \varphi) \quad (5.33)$$

$$+ \bigoplus_{X,Y,Z} -g(\Phi^{-1}\nabla_Y \Phi(\nabla \varphi), \Phi^{-1}\nabla_Z \Phi(X)) + g(\Phi^{-1}\nabla_Y \Phi(X), \Phi^{-1}\nabla_Z \Phi(\nabla \varphi)) \quad (5.34)$$

$$+ \bigoplus_{X,Y,Z} \frac{1}{2} \|\nabla \varphi\|^2 (\Phi_X Y Z - \Phi_Y X Z) - \frac{1}{4} \|\nabla \varphi\|^2 (\varphi_Y g(Z, X) - \varphi_Z g(Y, X)). \quad (5.35)$$

Moreover

$$(5.34) = \bigoplus_{X,Y,Z} -g(\Phi^{-1}\nabla_X \Phi(\nabla \varphi), \Phi^{-1}d\Phi(Y, Z)).$$

Since $\bigoplus_{X,Y,Z} \Phi_Z i \nabla \varphi \Phi_X i Y = \bigoplus_{X,Y,Z} \Phi_Y i \nabla \varphi \Phi_Z i X$, then

$$\begin{aligned} (5.31) &= \bigoplus_{X,Y,Z} \sum_i -\Phi_X i \nabla \varphi (\Phi_Y i Z - \Phi_Z i Y) + \Phi_Y i \nabla \varphi (\Phi_X i Z - \Phi_Z i X) \\ &= \bigoplus_{X,Y,Z} \sum_i -2\Phi_X i \nabla \varphi (\Phi_Y i Z - \Phi_Z i Y) \\ &= \bigoplus_{X,Y,Z} \sum_i -2\Phi_X i \nabla \varphi g(\Phi^{-1}d\Phi(Z, Y), e_i) - 2\varphi_Y \Phi_X Z \nabla \varphi + 2\varphi_Z \Phi_X Y \nabla \varphi \\ &= \bigoplus_{X,Y,Z} \sum_i -2\Phi_X i \nabla \varphi g(\Phi^{-1}d\Phi(Z, Y), e_i) - 2\varphi_Y g(\Phi^{-1}d\Phi(X, Z), \nabla \varphi) \\ &= \bigoplus_{X,Y,Z} \sum_i -2(-\Phi_X \nabla \varphi i + \varphi_X \varphi_i) g(\Phi^{-1}d\Phi(Z, Y), e_i) - 2\varphi_Y g(\Phi^{-1}d\Phi(X, Z), \nabla \varphi) \\ &= \bigoplus_{X,Y,Z} 2g(\Phi^{-1}d\Phi(Z, Y), \Phi^{-1}\nabla_X \Phi(\nabla \varphi)) - 2\varphi_X g(\Phi^{-1}d\Phi(Z, Y), \nabla \varphi) - 2\varphi_Y g(\Phi^{-1}d\Phi(X, Z), \nabla \varphi) \\ &= \bigoplus_{X,Y,Z} -2g(\Phi^{-1}\nabla_X \Phi(\nabla \varphi), \Phi^{-1}d\Phi(Y, Z)) + 4\varphi_X g(\Phi^{-1}d\Phi(Y, Z), \nabla \varphi). \end{aligned}$$

We have $\Phi_X Y Z - \Phi_Y X Z = g(\Phi^{-1}d\Phi(X, Y), Z)$ and $\bigoplus_{X,Y,Z} \frac{1}{4} \|\nabla \varphi\|^2 (\varphi_X g(Y, Z) - \varphi_Z g(Y, X)) = 0$, that we will replace in (5.35). Moreover

$$(5.30) + (5.32) = \bigoplus_{X,Y,Z} \sum_{ij} (\Phi_X ij - \Phi_X ji) g(\Phi^{-1}\nabla_Y \Phi(e_i), \Phi^{-1}\nabla_Z \Phi(e_j)). \quad (5.36)$$

Therefore,

$$\begin{aligned} (5.21) + (5.22) &= \bigoplus_{X,Y,Z} \sum_{ij} (\Phi_X ij - \Phi_X ji) g(\Phi^{-1}\nabla_Y \Phi(e_i), \Phi^{-1}\nabla_Z \Phi(e_j)) \\ &+ \bigoplus_{X,Y,Z} -3g(\Phi^{-1}\nabla_X \Phi(\nabla \varphi), \Phi^{-1}d\Phi(Y, Z)) + 3g(\varphi_X \nabla \varphi, \Phi^{-1}d\Phi(Y, Z)) \\ &+ \bigoplus_{X,Y,Z} \frac{1}{2} \|\nabla \varphi\|^2 g(X, \Phi^{-1}d\Phi(Y, Z)). \quad \square \end{aligned}$$

Proposition 5.5.

$$\langle S' \wedge (\hat{R}^M - \frac{1}{2}dS' - \frac{1}{3}(S')^2) \rangle_{\hat{g}}(X, Y, Z) = \quad (5.37)$$

$$= \bigoplus_{X, Y, Z} \frac{1}{4} \langle \Phi^{-1}R^{\perp}(Y, Z) + R^M(Y, Z), \Phi^{-1}\nabla_X\Phi - \frac{1}{4}d(g(\Phi^{-1}d\Phi(\cdot, \cdot), \nabla\varphi)) \rangle(X, Y, Z) \quad (5.38)$$

$$+ \bigoplus_{X, Y, Z} \frac{1}{12} \langle \Phi^{-1}\nabla_X\Phi, [\Phi^{-1}\nabla_Y\Phi, \Phi^{-1}\nabla_Z\Phi] \rangle. \quad (5.39)$$

Proof. Let e_i a g -orthonormal frame of TM . Then \hat{e}_i is a \hat{g} -orthonormal frame. We have

$$\langle S' \wedge (\hat{R} - \frac{1}{2}dS' - \frac{1}{3}(S')^2) \rangle_{\hat{g}}(X, Y, Z) = \bigoplus_{X, Y, Z} \langle S'(X), (\hat{R} - \frac{1}{2}dS' - \frac{1}{3}(S')^2)(Y, Z) \rangle_{\hat{g}} = \quad (5.40)$$

$$= \bigoplus_{X, Y, Z} \sum_{ij} \frac{1}{2} \langle S'(X), \hat{e}_i \wedge \hat{e}_j \rangle_{\hat{g}} \left(\hat{R}(Y, Z, \hat{e}_i, \hat{e}_j) - \frac{1}{2} \langle dS'(Y \wedge Z), \hat{e}_i \wedge \hat{e}_j \rangle_{\hat{g}} - \frac{1}{3} \langle (S')^2(Y \wedge Z), \hat{e}_i \wedge \hat{e}_j \rangle_{\hat{g}} \right)$$

$$= \bigoplus_{X, Y, Z} \sum_{ij} \frac{1}{2} \hat{g}(S'(X, \hat{e}_i), \hat{e}_j) \left(hR^M(Y, Z, \hat{e}_i, \hat{e}_j) + h\phi \bullet g(Y, Z, \hat{e}_i, \hat{e}_j) - \frac{1}{2} \langle dS'(Y \wedge Z), \hat{e}_i \wedge \hat{e}_j \rangle_{\hat{g}} \right) \\ - \frac{1}{6} \hat{g}(S'(X, \hat{e}_i), \hat{e}_j) (\hat{g}(S'(Y, \hat{e}_i), S'(Z, \hat{e}_j)) - \hat{g}(S'(Y, \hat{e}_j), S'(Z, \hat{e}_i))) \quad (5.41)$$

$$= \bigoplus_{X, Y, Z} \sum_{ij} \frac{1}{2} g(S'(X, e_i), e_j) \left(R^M(Y, Z, e_i, e_j) + \phi \bullet g(Y, Z, e_i, e_j) \right) \\ - \frac{1}{4} g(S'(X, e_i), e_j) (\langle \hat{\nabla}_Y S(Z) - \hat{\nabla}_Z S(Y), \hat{e}_i \wedge \hat{e}_j \rangle_{\hat{g}}) \quad (5.42)$$

$$- \frac{1}{6} g(S'(X, e_i), e_j) (g(S'(Y, e_i), S'(Z, e_j)) - g(S'(Y, e_j), S'(Z, e_i))). \quad (5.43)$$

We assume that at a given point p_0 , $\hat{\nabla}X = \hat{\nabla}Y = \hat{\nabla}Z = \hat{\nabla}\hat{e}_i = 0$. Thus, at p_0 , $\nabla_X Y = -\hat{S}(X, Y) = -\frac{1}{2}\varphi_X Y - \frac{1}{2}\varphi_Y X + \frac{1}{2}g(X, Y)\nabla\varphi$, and similarly for the other vector fields. The following computations are computed at p_0 .

$$d(g(X, Y))(Z) = -\varphi_Z g(X, Y) \quad (5.44)$$

$$d(\varphi_X)(Y) = Hess\varphi(X, Y) - \varphi_X \varphi_Y + \frac{1}{2} \|\nabla\varphi\|^2 g(X, Y) \quad (5.45)$$

and since $\bar{R}(X, Y)\Phi = -\nabla_{X,Y}^2\Phi + \nabla_{Y,X}^2\Phi$, we have

$$d(\Phi_X ZW)(Y) - d(\Phi_Y ZW)(X) = \quad (5.46)$$

$$= -2\varphi_Y \Phi_X ZW + 2\varphi_X \Phi_Y ZW + g(\Phi^{-1}(\bar{R}(X, Y)\Phi)(Z), W) \\ - \frac{1}{2} \varphi_Z g(\Phi^{-1}d\Phi(X, Y), W) + \frac{1}{2} g(Y, Z) \Phi_X \nabla\varphi W - \frac{1}{2} g(X, Z) \Phi_Y \nabla\varphi W \\ + g(\Phi^{-1}\nabla_X\Phi(Z), \Phi^{-1}\nabla_Y\Phi(W)) - g(\Phi^{-1}\nabla_Y\Phi(Z), \Phi^{-1}\nabla_X\Phi(W)) \\ - \frac{1}{2} \varphi_W \Phi_X ZY + \frac{1}{2} g(Y, W) \Phi_X Z\nabla\varphi + \frac{1}{2} \varphi_W \Phi_Y ZX - \frac{1}{2} g(X, W) \Phi_Y Z\nabla\varphi.$$

Applying eqs (5.44), (5.45), (5.46) and (5.17) we get

$$(5.42) = \bigoplus_{X, Y, Z} \sum_{ij} -\frac{1}{4} g(S'(X, e_i), e_j) \left(\nabla_Y (\langle S'(Z), \hat{e}_i \wedge \hat{e}_j \rangle_{\hat{g}}) - \nabla_Z (\langle S'(Y), \hat{e}_i \wedge \hat{e}_j \rangle_{\hat{g}}) \right) \\ = \bigoplus_{X, Y, Z} \sum_{ij} -\frac{1}{4} g(S'(X, e_i), e_j) \left(\nabla_Y (g(S'(Z, e_i), e_j)) - \nabla_Z (g(S'(Y, e_i), e_j)) \right) \\ = \bigoplus_{X, Y, Z} \sum_{ij} -\frac{1}{4} \left(\Phi_X ij - \frac{1}{2} \delta_{ij} \varphi_X - \frac{1}{2} \varphi_i g(X, e_j) + \frac{1}{2} g(X, e_i) \varphi_j \right). \quad (5.47)$$

$$\cdot \left(-2\varphi_Y \Phi_Z ij + 2\varphi_Z \Phi_Y ij + g(\Phi^{-1}(\bar{R}(Z, Y)\Phi)(e_i), e_j) \right) \quad (5.48)$$

$$- \frac{1}{2} \varphi_i g(\Phi^{-1}d\Phi(Z, Y), e_j) + \frac{1}{2} g(Y, e_i) \Phi_Z \nabla\varphi_j - \frac{1}{2} g(Z, e_i) \Phi_Y \nabla\varphi_j \quad (5.49)$$

$$+ g(\Phi^{-1}\nabla_Z\Phi(e_i), \Phi^{-1}\nabla_Y\Phi(e_j)) - g(\Phi^{-1}\nabla_Y\Phi(e_i), \Phi^{-1}\nabla_Z\Phi(e_j)) \quad (5.50)$$

$$- \frac{1}{2} \varphi_j \Phi_Z iY + \frac{1}{2} g(Y, e_j) \Phi_Z i\nabla\varphi + \frac{1}{2} \varphi_j \Phi_Y iZ - \frac{1}{2} g(Z, e_j) \Phi_Y i\nabla\varphi \quad (5.51)$$

$$- \frac{1}{8} g(S'(X, e_i), e_j) \cdot \left(-Hess\varphi + 2d\varphi \otimes d\varphi - \frac{1}{2} \|\nabla\varphi\|^2 g \right) \bullet g(Y, Z, e_i, e_j) \quad (5.52)$$

$$(5.47) + \dots + (5.51) =$$

$$= \bigoplus_{x,Y,Z} \sum_{ij} (\frac{1}{2} \varphi_Y \Phi_X ij \Phi_Z ij - \frac{1}{2} \varphi_Z \Phi_X ij \Phi_Y ij) \quad (5.53)$$

$$+ \bigoplus_{x,Y,Z} -\frac{1}{4} \langle \Phi^{-1} \bar{R}(Z, Y) \Phi, \Phi^{-1} \nabla_X \Phi \rangle + \frac{1}{8} \langle \Phi^{-1} \nabla_X \Phi (\nabla \varphi), \Phi^{-1} d\Phi(Z, Y) \rangle \quad (5.54)$$

$$+ \bigoplus_{x,Y,Z} -\frac{1}{8} \langle \Phi^{-1} \nabla_X \Phi (Y), \Phi^{-1} \nabla_Z \Phi (\nabla \varphi) \rangle + \frac{1}{8} \langle \Phi^{-1} \nabla_X \Phi (Z), \Phi^{-1} \nabla_Y \Phi (\nabla \varphi) \rangle \quad (5.55)$$

$$+ \bigoplus_{x,Y,Z} \sum_{ij} -\frac{1}{4} \Phi_X ij g(\Phi^{-1} \nabla_Z \Phi (e_i), \Phi^{-1} \nabla_Y \Phi (e_j)) + \frac{1}{4} \Phi_X ij g(\Phi^{-1} \nabla_Y \Phi (e_i), \Phi^{-1} \nabla_Z \Phi (e_j)) \quad (5.56)$$

$$+ \bigoplus_{x,Y,Z} \sum_i \frac{1}{8} \Phi_X i \nabla \varphi \Phi_Z i Y - \frac{1}{8} \Phi_X i Y \Phi_Z i \nabla \varphi - \frac{1}{8} \Phi_X i \nabla \varphi \Phi_Y i Z + \frac{1}{8} \Phi_X i Z \Phi_Y i \nabla \varphi \quad (5.57)$$

$$+ \bigoplus_{x,Y,Z} -\frac{1}{4} \varphi_Y \Phi_Z \nabla \varphi X + \frac{1}{4} \varphi_Z \Phi_Y \nabla \varphi X + \frac{1}{4} \varphi_Y \Phi_Z X \nabla \varphi - \frac{1}{4} \varphi_Z \Phi_Y X \nabla \varphi$$

$$+ \bigoplus_{x,Y,Z} +\frac{1}{8} g(\Phi^{-1} (\bar{R}(Z, Y) \Phi) (\nabla \varphi), X) - \frac{1}{8} g(\Phi^{-1} (\bar{R}(Z, Y) \Phi) (X), \nabla \varphi)$$

$$+ \bigoplus_{x,Y,Z} -\frac{1}{16} \|\nabla \varphi\|^2 g(\Phi^{-1} d\Phi(Z, Y), X) + \frac{1}{16} \varphi_X g(\Phi^{-1} d\Phi(Z, Y), \nabla \varphi)$$

$$+ \bigoplus_{x,Y,Z} +\frac{1}{16} \varphi_Y \Phi_Z \nabla \varphi X - \frac{1}{16} \varphi_Z \Phi_Y \nabla \varphi X - \frac{1}{16} g(Y, X) \Phi_Z \nabla \varphi \nabla \varphi + \frac{1}{16} g(Z, X) \Phi_Y \nabla \varphi \nabla \varphi \quad (5.58)$$

$$+ \bigoplus_{x,Y,Z} \frac{1}{8} g(\Phi^{-1} \nabla_Z \Phi (\nabla \varphi), \Phi^{-1} \nabla_Y \Phi (X)) - \frac{1}{8} g(\Phi^{-1} \nabla_Y \Phi (\nabla \varphi), \Phi^{-1} \nabla_Z \Phi (X)) \quad (5.59)$$

$$+ \bigoplus_{x,Y,Z} -\frac{1}{8} g(\Phi^{-1} \nabla_Z \Phi (X), \Phi^{-1} \nabla_Y \Phi (\nabla \varphi)) + \frac{1}{8} g(\Phi^{-1} \nabla_Y \Phi (X), \Phi^{-1} \nabla_Z \Phi (\nabla \varphi)) \quad (5.60)$$

$$+ \bigoplus_{x,Y,Z} -\frac{1}{16} \varphi_X \Phi_Z \nabla \varphi Y + \frac{1}{16} g(Y, X) \Phi_Z \nabla \varphi \nabla \varphi + \frac{1}{16} \varphi_X \Phi_Y \nabla \varphi Z - \frac{1}{16} g(Z, X) \Phi_Y \nabla \varphi \nabla \varphi \quad (5.61)$$

$$+ \bigoplus_{x,Y,Z} \frac{1}{16} \|\nabla \varphi\|^2 \Phi_Z X Y - \frac{1}{16} \varphi_Y \Phi_Z X \nabla \varphi - \frac{1}{16} \|\nabla \varphi\|^2 \Phi_Y X Z + \frac{1}{16} \varphi_Z \Phi_Y X \nabla \varphi. \quad (5.62)$$

Note that by (5.29), (5.53) = 0, and the second term of (5.54) is equal to (5.55) = (5.59) = (5.60). We also have using (5.14)(5.16)

$$\begin{aligned} (5.56) &= \bigoplus_{x,Y,Z} \sum_i \frac{1}{8} (-\Phi_X \nabla \varphi i + \varphi_X \varphi_i) (\Phi_Z i Y - \Phi_Y i Z) \\ &\quad - \frac{1}{8} (-\Phi_X Y i + \varphi_X g(i, Y)) (-\Phi_Z \nabla \varphi i + \varphi_Z \varphi_i) + \frac{1}{8} (-\Phi_X Z i + \varphi_X g(i, Z)) (-\Phi_Y \nabla \varphi i + \varphi_Y \varphi_i) \\ &= \bigoplus_{x,Y,Z} -\frac{1}{8} g(\Phi^{-1} \nabla_X \Phi (\nabla \varphi), \Phi^{-1} d\Phi(Y, Z)) + \frac{1}{8} \varphi_X g(\Phi^{-1} d\Phi(Y, Z), \nabla \varphi) \\ &\quad - \frac{1}{8} g(\Phi^{-1} \nabla_X \Phi (Y), \Phi^{-1} \nabla_Z \Phi (\nabla \varphi)) + \frac{1}{8} g(\Phi^{-1} \nabla_X \Phi (Z), \Phi^{-1} \nabla_Y \Phi (\nabla \varphi)) \\ &\quad + \frac{1}{8} \varphi_Z (-\Phi_X \nabla \varphi Y + \Phi_X Y \nabla \varphi) + \frac{1}{8} \varphi_Y (\Phi_X \nabla \varphi Z - \Phi_X Z \nabla \varphi) + \frac{1}{8} \varphi_X (\Phi_Z \nabla \varphi Y - \Phi_Y \nabla \varphi Z) \\ &= \bigoplus_{x,Y,Z} -\frac{1}{4} g(\Phi^{-1} \nabla_X \Phi (\nabla \varphi), \Phi^{-1} d\Phi(Y, Z)) + \frac{1}{8} \varphi_X g(\Phi^{-1} d\Phi(Y, Z), \nabla \varphi) \\ &\quad + \bigoplus_{x,Y,Z} +\frac{1}{8} \varphi_Z (-\Phi_X \nabla \varphi Y + \Phi_X Y \nabla \varphi) + \frac{1}{8} \varphi_Y (\Phi_X \nabla \varphi Z - \Phi_X Z \nabla \varphi) + \frac{1}{8} \varphi_X (\Phi^{-1} d\Phi(Y, Z), \nabla \varphi). \end{aligned}$$

And using again (5.16)

$$\begin{aligned}
(5.57) + (5.58) + (5.61) + (5.62) &= \\
&= \bigoplus_{x,y,z} \frac{3}{16} \varphi_Y (-\Phi_Z \nabla \varphi X + \Phi_Z X \nabla \varphi) + \frac{3}{16} \varphi_Z (\Phi_Y \nabla \varphi X - \Phi_Y X \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} \frac{1}{16} \varphi_X (-\Phi_Z \nabla \varphi Y + \Phi_Y \nabla \varphi Z) + \frac{1}{16} \|\nabla \varphi\|^2 g(\Phi^{-1} d\Phi(Y, Z), X) \\
&= \bigoplus_{x,y,z} \frac{3}{16} \varphi_Y (-\Phi_X \nabla \varphi Z - g(\Phi^{-1} d\Phi(X, Z), \nabla \varphi) - g(\varphi_Z X - \varphi_X Z, \nabla \varphi) + g(\Phi^{-1} d\Phi(Z, X), \nabla \varphi) + \Phi_X Z \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} \frac{3}{16} \varphi_Z (\Phi_X \nabla \varphi Y + g(\Phi^{-1} d\Phi(X, Y), \nabla \varphi) + g(\varphi_Y X - \varphi_X Y, \nabla \varphi) - g(\Phi^{-1} d\Phi(Y, X), \nabla \varphi) - \Phi_X Y \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} \frac{1}{16} \varphi_X (-\Phi_Z \nabla \varphi Y + \Phi_Y \nabla \varphi Z) + \frac{1}{16} \|\nabla \varphi\|^2 g(X, \Phi^{-1} d\Phi(Y, Z)) \\
&= \bigoplus_{x,y,z} \frac{3}{16} \varphi_Y (-\Phi_X \nabla \varphi Z + \Phi_X Z \nabla \varphi) + \frac{3}{8} \varphi_Y g(\Phi^{-1} d\Phi(Z, X), \nabla \varphi) + \frac{3}{16} \varphi_Z (\Phi_X \nabla \varphi Y - \Phi_X Y \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} \frac{3}{8} \varphi_Z g(\Phi^{-1} d\Phi(X, Y), \nabla \varphi) + \frac{1}{16} \varphi_X (\Phi^{-1} d\Phi(Z, Y), \nabla \varphi) + \frac{1}{16} \|\nabla \varphi\|^2 g(X, \Phi^{-1} d\Phi(Y, Z)) \\
&= \bigoplus_{x,y,z} \frac{3}{16} \varphi_Y (-\Phi_X \nabla \varphi Z + \Phi_X Z \nabla \varphi) + \frac{3}{16} \varphi_Z (\Phi_X \nabla \varphi Y - \Phi_X Y \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} \frac{11}{16} \varphi_X (\Phi^{-1} d\Phi(Y, Z), \nabla \varphi) + \frac{1}{16} \|\nabla \varphi\|^2 g(X, \Phi^{-1} d\Phi(Y, Z)).
\end{aligned}$$

Thus,

$$\begin{aligned}
(5.47) + \dots + (5.51) &= \\
&= \bigoplus_{x,y,z} -\frac{1}{4} \langle \Phi^{-1} \bar{R}(Z, Y) \Phi, \Phi^{-1} \nabla_X \Phi \rangle + \frac{1}{2} \langle \Phi^{-1} \nabla_X \Phi (\nabla \varphi), \Phi^{-1} d\Phi(Z, Y) \rangle \\
&\quad + \bigoplus_{x,y,z} \sum_{ij} \frac{1}{4} (\Phi_X ij - \Phi_X ji) g(\Phi^{-1} \nabla_Y \Phi(e_i), \Phi^{-1} \nabla_Z \Phi(e_j)) \\
&\quad + \bigoplus_{x,y,z} -\frac{1}{4} g(\Phi^{-1} \nabla_X \Phi (\nabla \varphi), \Phi^{-1} d\Phi(Y, Z)) + \frac{1}{8} \varphi_X g(\Phi^{-1} d\Phi(Y, Z), \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} +\frac{1}{8} \varphi_Z (-\Phi_X \nabla \varphi Y + \Phi_X Y \nabla \varphi) + \frac{1}{8} \varphi_Y (\Phi_X \nabla \varphi Z - \Phi_X Z \nabla \varphi) + \frac{1}{8} \varphi_X g(\Phi^{-1} d\Phi(Y, Z), \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} +\frac{1}{8} g(\Phi^{-1} (\bar{R}(Z, Y) \Phi) (\nabla \varphi), X) - \frac{1}{8} g(\Phi^{-1} (\bar{R}(Z, Y) \Phi) (X), \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} -\frac{1}{16} \|\nabla \varphi\|^2 g(\Phi^{-1} d\Phi(Z, Y), X) + \frac{1}{16} \varphi_X g(\Phi^{-1} d\Phi(Z, Y), \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} \frac{3}{16} \varphi_Y (-\Phi_X \nabla \varphi Z + \Phi_X Z \nabla \varphi) + \frac{3}{16} \varphi_Z (\Phi_X \nabla \varphi Y - \Phi_X Y \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} \frac{11}{16} \varphi_X g(\Phi^{-1} d\Phi(Y, Z), \nabla \varphi) + \frac{1}{16} \|\nabla \varphi\|^2 g(X, \Phi^{-1} d\Phi(Y, Z)) \\
&= \bigoplus_{x,y,z} -\frac{1}{4} \langle \Phi^{-1} (\bar{R}(Z, Y) \Phi), \Phi^{-1} \nabla_X \Phi \rangle - \frac{3}{4} \langle \Phi^{-1} \nabla_X \Phi (\nabla \varphi), \Phi^{-1} d\Phi(Y, Z) \rangle \\
&\quad + \bigoplus_{x,y,z} \sum_{ij} \frac{1}{4} (\Phi_X ij - \Phi_X ji) g(\Phi^{-1} \nabla_Y \Phi(e_i), \Phi^{-1} \nabla_Z \Phi(e_j)) + \frac{7}{8} \varphi_X g(\Phi^{-1} d\Phi(Y, Z), \nabla \varphi) \\
&\quad + \bigoplus_{x,y,z} +\frac{1}{4} g(\Phi^{-1} (\bar{R}(Y, Z) \Phi) X, \nabla \varphi) - \frac{1}{8} \|\nabla \varphi\|^2 g(\Phi^{-1} d\Phi(Z, Y), X) \\
&\quad + \bigoplus_{x,y,z} \frac{1}{16} \varphi_Y (-\Phi_X \nabla \varphi Z + \Phi_X Z \nabla \varphi) + \frac{1}{16} \varphi_Z (\Phi_X \nabla \varphi Y - \Phi_X Y \nabla \varphi). \tag{5.63}
\end{aligned}$$

Now,

$$\begin{aligned}
(5.63) &= \bigoplus_{x,y,z} \frac{1}{16} \varphi_Y (\Phi_X Z \nabla \varphi - \varphi_X \varphi_Z + \Phi_X Z \nabla \varphi) + \frac{1}{16} \varphi_Z (-\Phi_X Y \nabla \varphi + \varphi_X \varphi_Y - \Phi_X Y \nabla \varphi) \\
&= \bigoplus_{x,y,z} \frac{1}{8} \varphi_Y \Phi_X Z \nabla \varphi - \frac{1}{8} \varphi_Z \Phi_X Y \nabla \varphi = \bigoplus_{x,y,z} -\frac{1}{8} \varphi_X g(\Phi^{-1} d\Phi(Y, Z), \nabla \varphi).
\end{aligned}$$

Finally using Lemma 5.5 with $\phi = \frac{1}{2}(-\frac{\|\nabla\varphi\|^2}{4}g + \frac{1}{2}d\varphi \otimes d\varphi - Hess\varphi)$

$$(5.52) + \bigoplus_{X,Y,Z} \sum_{ij} \frac{1}{2}g(S'(X, e_i), e_j)\phi \bullet g(Y, Z, e_i, e_j) = \bigoplus_{X,Y,Z} -\frac{1}{4}(Hess\varphi + d\varphi \otimes d\varphi)(X, \Phi^{-1}d\Phi(Y, Z)). \quad (5.64)$$

Therefore from (5.41), (5.42), (5.43), and Lemma 5.6

$$\begin{aligned} \langle S' \wedge (\hat{R} - \frac{1}{2}dS' - \frac{1}{3}(S')^2) \rangle_{\hat{g}}(X, Y, Z) &= \\ &= \bigoplus_{X,Y,Z} \sum_i -\frac{1}{2}R^M(Y, Z, \Phi^{-1}\nabla_X\Phi(e_i), e_i) + (5.47) + (5.48) + (5.49) + (5.50) + (5.51) + (5.43) + (5.64) \\ &= \bigoplus_{X,Y,Z} \sum_i -\frac{1}{2}R^M(Y, Z, \Phi^{-1}\nabla_X\Phi(e_i), e_i) - \frac{1}{4}\langle \Phi^{-1}\bar{R}(Z, Y)\Phi, \Phi^{-1}\nabla_X\Phi \rangle \\ &\quad + \bigoplus_{X,Y,Z} -\frac{3}{4}\langle \Phi^{-1}\nabla_X\Phi(\nabla\varphi), \Phi^{-1}d\Phi(Y, Z) \rangle + \sum_{ij} \frac{1}{4}(\Phi_X ij - \Phi_X ji)g(\Phi^{-1}\nabla_Y\Phi(e_i), \Phi^{-1}\nabla_Z\Phi(e_j)) \\ &\quad + \bigoplus_{X,Y,Z} +\frac{6}{8}\varphi_X g(\Phi^{-1}d\Phi(Y, Z), \nabla\varphi) + \frac{1}{4}g(\Phi^{-1}(\bar{R}(Y, Z)\Phi)(X), \nabla\varphi) - \frac{1}{8}\|\nabla\varphi\|^2 g(\Phi^{-1}d\Phi(Z, Y), X) \\ &\quad + \bigoplus_{X,Y,Z} -\frac{1}{4}Hess\varphi(X, \Phi^{-1}d\Phi(Y, Z)) - \frac{1}{4}\varphi_X g(\Phi^{-1}d\Phi(Y, Z), \nabla\varphi) \\ &\quad + \bigoplus_{X,Y,Z} \sum_{ij} -\frac{1}{6}(\Phi_X ij - \Phi_X ji)g(\Phi^{-1}\nabla_Y\Phi(e_i), \Phi^{-1}\nabla_Z\Phi(e_j)) + \frac{1}{2}g(\Phi^{-1}\nabla_X\Phi(\nabla\varphi), \Phi^{-1}d\Phi(Y, Z)) \\ &\quad + \bigoplus_{X,Y,Z} -\frac{1}{4}\varphi_X g(\Phi^{-1}d\Phi(Y, Z), \nabla\varphi) - \frac{1}{8}\|\nabla\varphi\|^2 g(X, \Phi^{-1}d\Phi(Y, Z)) \\ &= \bigoplus_{X,Y,Z} \sum_i -\frac{1}{2}R^M(Y, Z, \Phi^{-1}\nabla_X\Phi(e_i), e_i) - \frac{1}{4}\langle \Phi^{-1}\bar{R}(Z, Y)\Phi, \Phi^{-1}\nabla_X\Phi \rangle + \frac{1}{4}g(\Phi^{-1}(\bar{R}(Y, Z)\Phi)(X), \nabla\varphi) \\ &\quad + \bigoplus_{X,Y,Z} -\frac{1}{4}\langle \Phi^{-1}\nabla_X\Phi(\nabla\varphi), \Phi^{-1}d\Phi(Y, Z) \rangle + \sum_{ij} \frac{1}{12}(\Phi_X ij - \Phi_X ji)g(\Phi^{-1}\nabla_Y\Phi(e_i), \Phi^{-1}\nabla_Z\Phi(e_j)) \\ &\quad + \bigoplus_{X,Y,Z} +\frac{1}{4}\varphi_X g(\Phi^{-1}d\Phi(Y, Z), \nabla\varphi) - \frac{1}{4}Hess\varphi(X, \Phi^{-1}d\Phi(Y, Z)). \end{aligned}$$

Note that

$$-\langle \Phi^{-1}\bar{R}(Z, Y)\Phi, \Phi^{-1}\nabla_X\Phi \rangle = \sum_i g(\Phi^{-1}R^\perp(Y, Z)\Phi(e_i), \Phi^{-1}\nabla_X\Phi(e_i)) - g(R^M(Y, Z)e_i, \Phi^{-1}\nabla_X\Phi(e_i)).$$

We have $\bigoplus_{X,Y,Z} g(\Phi^{-1}(\bar{R}(Y, Z)\Phi)(X), \nabla\varphi) = \bigoplus_{X,Y,Z} g(\Phi^{-1}(R^\perp(Y, Z)\Phi(X)), \nabla\varphi)$, for R^M satisfies Bianchi equality, and recall that $\bigoplus_{X,Y,Z} \Phi^{-1}(R^\perp(Y, Z)\Phi(X)) = -\Phi^{-1}d^2\Phi(X, Y, Z)$. Now,

$$\begin{aligned} &\bigoplus_{X,Y,Z} \sum_{ij} (\Phi_X ij - \Phi_X ji)g(\Phi^{-1}\nabla_Y\Phi(e_i), \Phi^{-1}\nabla_Z\Phi(e_j)) = \\ &= \bigoplus_{X,Y,Z} \langle \Phi^{-1}\nabla_Y\Phi, (\Phi^{-1}\nabla_Z\Phi) \circ (\Phi^{-1}\nabla_X\Phi) \rangle - \langle (\Phi^{-1}\nabla_Y\Phi) \circ (\Phi^{-1}\nabla_X\Phi), \Phi^{-1}\nabla_Z\Phi \rangle \\ &= \bigoplus_{X,Y,Z} \langle \Phi^{-1}\nabla_X\Phi, (\Phi^{-1}\nabla_Y\Phi) \circ (\Phi^{-1}\nabla_Z\Phi) - (\Phi^{-1}\nabla_Z\Phi) \circ (\Phi^{-1}\nabla_Y\Phi) \rangle. \end{aligned}$$

Thus

$$\begin{aligned} \langle S' \wedge (\hat{R} - \frac{1}{2}dS' - \frac{1}{3}(S')^2) \rangle_{\hat{g}}(X, Y, Z) &= \\ &= \bigoplus_{X,Y,Z} \frac{1}{4}\langle \Phi^{-1}R^\perp(Y, Z)\Phi + R^M(Y, Z), \Phi^{-1}\nabla_X\Phi \rangle + \frac{1}{4}g(\Phi^{-1}(R^\perp(Y, Z)\Phi(X)), \nabla\varphi) \\ &\quad + \bigoplus_{X,Y,Z} -\frac{1}{4}g(\Phi^{-1}\nabla_X\Phi(\nabla\varphi), \Phi^{-1}d\Phi(Y, Z)) + \frac{1}{12}\langle \Phi^{-1}\nabla_X\Phi, [\Phi^{-1}\nabla_Y\Phi, \Phi^{-1}\nabla_Z\Phi] \rangle \\ &\quad + \bigoplus_{X,Y,Z} +\frac{1}{4}(d\varphi \otimes d\varphi - Hess\varphi)(X, \Phi^{-1}d\Phi(Y, Z)). \end{aligned}$$

Using (5.13)

$$\bigoplus_{X,Y,Z} g(\Phi^{-1}\nabla_X\Phi(\nabla\varphi), \Phi^{-1}d\Phi(Y,Z)) = \bigoplus_{X,Y,Z} -g(\Phi^{-1}\nabla_X\Phi(\Phi^{-1}d\Phi(Y,Z)), \nabla\varphi) + \varphi_X g(\Phi^{-1}d\Phi(Y,Z), \nabla\varphi)$$

with $\varphi_X g(\Phi^{-1}d\Phi(Y,Z), \nabla\varphi) = (d\varphi \otimes d\varphi)(X, \Phi^{-1}d\Phi(Y,Z))$. Since $\nabla_X\Phi^{-1} = -\Phi^{-1}(\nabla_X\Phi)\Phi^{-1}$, we have

$$-\Phi^{-1}d^2\Phi(X,Y,Z) = -d(\Phi^{-1}d\Phi)(X,Y,Z) + \bigoplus_{X,Y,Z} -\Phi^{-1}\nabla_X\Phi(\Phi^{-1}d\Phi(Y,Z)).$$

Since $b(\Phi(R^\perp)) = -\Phi^{-1}d^2\Phi$ we obtain

$$\begin{aligned} \bigoplus_{X,Y,Z} g(\Phi^{-1}(R^\perp(Y,Z)\Phi(X)), \nabla\varphi) &= -g(d(\Phi^{-1}d\Phi)(X,Y,Z), \nabla\varphi) - \bigoplus_{X,Y,Z} g(\Phi^{-1}\nabla_X\Phi(\Phi^{-1}d\Phi(Y,Z)), \nabla\varphi) = \\ &= -g(d(\Phi^{-1}d\Phi)(X,Y,Z), \nabla\varphi) + \bigoplus_{X,Y,Z} g(\Phi^{-1}\nabla_X\Phi(\nabla\varphi), \Phi^{-1}d\Phi(Y,Z)) - (d\varphi \otimes d\varphi)(X, \Phi^{-1}d\Phi(Y,Z)) \end{aligned}$$

Thus,

$$\langle \mathcal{S}' \wedge (\hat{R}^M - \tfrac{1}{2}d\mathcal{S}' - \tfrac{1}{3}(\mathcal{S}')^2) \rangle_{\hat{g}}(X,Y,Z) = \quad (5.65)$$

$$= \bigoplus_{X,Y,Z} \tfrac{1}{4} \langle \Phi^{-1}R^\perp(Y,Z) + R^M(Y,Z), \Phi^{-1}\nabla_X\Phi \rangle - \tfrac{1}{4}g(d(\Phi^{-1}d\Phi)(X,Y,Z), \nabla\varphi) \quad (5.66)$$

$$+ \bigoplus_{X,Y,Z} \tfrac{1}{12} \langle \Phi^{-1}\nabla_X\Phi, [\Phi^{-1}\nabla_Y\Phi, \Phi^{-1}\nabla_Z\Phi] \rangle - \tfrac{1}{4}Hess\varphi(X, \Phi^{-1}d\Phi(Y,Z)). \quad (5.67)$$

Finally $d(g(\Phi^{-1}d\Phi(\cdot, \cdot), \nabla\varphi))(X,Y,Z) = g(d(\Phi^{-1}d\Phi)(X,Y,Z), \nabla\varphi) + \bigoplus_{X,Y,Z} Hess\varphi(X, \Phi^{-1}d\Phi(Y,Z))$

what proves the Proposition. \square

Proposition 5.6. *If $F : M \rightarrow N$ is a non- J -holomorphic Cayley submanifold and N is Ricci-flat then (1.6) – (1.7) holds.*

Proof. To prove (1.6) we note that from (1.5) and (5.1) and Corollary 5.1

$$p_1(\wedge_-^2 NM) = p_1(\wedge_-^2 TM) + 4(\mathcal{X}(M) - \mathcal{X}(NM)) = p_1(\wedge_-^2 TM) + 2(p_1(NM) - p_1(M)).$$

Since $R'(X,Y,Z,W) = R^\perp(X,Y,\Phi(Z),\Phi(W))$ and $\Phi : (TM, \hat{g}, \nabla') \rightarrow (NM, g, \nabla^\perp)$ is a parallel isometry along $M \sim \mathcal{C}$, then on this open set $p_1(NM) = p_1(R^\perp) = p_1(R')$, as forms defined by the formulas (5.3). From (5.11), Proposition 5.5, and that $d(g(\Phi^{-1}d\Phi(\cdot, \cdot), \nabla\varphi)) = 0$ we obtain (1.6)-(1.7). \square

Proposition 5.7. *If $(Y,Z) \rightarrow g(\nabla_X\Phi(Y), \Phi(Z))$ is symmetric then $p_1(\wedge_-^2 NM) = p_1(\wedge_-^2 TM)$.*

Proof. From Proposition 5.2, $R^\perp(X,Y,\Phi(\hat{Z}),\Phi(\hat{W})) = R^M(X,Y,Z,W)$ and so the characteristic classes induced by (R^\perp, g) are the same as the ones induced by (R^M, g) . We could also check directly from $\Phi_A BC = \tfrac{1}{2}\varphi_{AG}(B,C)$ that all terms of η in (1.6) i.e (5.38)-(5.39) vanish. \square

5.4 Homogeneous complex points

Let us assume that $F : M \rightarrow N$ is a compact Cayley submanifold, and let η be the 3-form on $M \sim \mathcal{C}$ defined as in (1.7). Since \mathcal{C} is the zero set of $\sin^2 \theta$, that has only zeros of finite order, this set has some regularity. Indeed, by the Malgrange's preparation theorem for smooth functions with zeros of finite order, locally we can find a coordinate chart $x = (x', x_4)$ onto an open set U of \mathbb{R}^4 such that $\sin^2 \theta$ can be written as $h(x) \left(\sum_{0 \leq a \leq k-1} w_a(x') x_4^a + x_4^k \right)$, where h never vanishes on U and w_a vanish of order $k - a$ at 0. Thus, the zero set of $\sin \theta$ can be locally parametrised as $\Sigma = \{x = (x', x_4) : \sum_{0 \leq a \leq k-1} w_a(x') x_4^a + x_4^k = 0\}$. This set represents the zeros of a polynomial function on the variable x_4 , with coefficients on the variable x' , so it is, in general, still quite complicate to handle. A simpler case is when we have a polynomial function of the type $(x_{d+1}^2 + \dots + x_4^2)^{k'} = 0$, as is it is the case with $k' = 1$, of x a Fermi coordinate chart of a submanifold Σ of dimension d .

Assume now f is a nonnegative continuous function defined on a open set V of M containing $\Sigma = f^{-1}(0)$ and smooth on $V \sim \Sigma$. For each $\epsilon > 0$ sufficiently small define

$$V_f(\Sigma, \epsilon) = \{q \in M : f(q) < \epsilon\}, \quad C_f(\Sigma, \epsilon) = \{q \in M : f(q) = \epsilon\}$$

For a dense set of regular values ϵ , $C_f(\Sigma, \epsilon)$ is a smooth hypersurface and is the boundary of $cl(V_f(\Sigma, \epsilon))$ and for each $q \in C_f(\Sigma, \epsilon)$, $T_q C_f(\Sigma, \epsilon) = [\nabla f(q)]^\perp$. If the decreasing sequence $d_M(V_f(\Sigma, \epsilon))$ converges to 0 when $\epsilon \rightarrow 0$, where d_M is the Lebesgue measure of M , then

$$\int_M d(\eta(\Phi)) = \lim_{\epsilon \rightarrow 0} \int_{M \sim V_f(\Sigma, \epsilon)} d(\eta(\Phi)) = - \lim_{\epsilon \rightarrow 0} \int_{C_f(\Sigma, \epsilon)} \eta(\Phi). \quad (5.68)$$

In case of Σ is a smooth hypersurface of M , f is smooth on V , and ∇f does not vanish on Σ , then for ϵ sufficiently small, $C_f(\Sigma, \epsilon)$ is connected, diffeomorphic to Σ , and converges (in the Lebesgue sense) to Σ and $M \sim V_f(\Sigma, \epsilon)$ to M when $\epsilon \rightarrow 0$. To see this let $\rho : M \rightarrow [0, 1]$ be a smooth map s.t. ρ values 1 on $V_f(\Sigma, 2r)$ and zero away from $V_f(\Sigma, 3r)$ for r sufficiently small, and let $\xi_t : M \rightarrow M$ be the one parameter family of diffeomorphisms generated by the vector field globally defined on M , $X_f = \rho \frac{\nabla f}{\|\nabla f\|^2}$. Then, there exist $\epsilon_0, \delta > 0$ such that $\forall |t| < \epsilon_0$ and $q \in V_f(\Sigma, \delta)$, $\xi_t(q) \in V_f(\Sigma, 2r)$, and so $\frac{\partial}{\partial t} f(\xi_t(q)) = df(\xi_t(q))(\frac{\partial}{\partial t} \xi_t(q)) = 1$ ([19]). That is $f(\xi_t(q)) = t + f(\xi_0(q)) = t + f(q)$. In particular $\forall 0 < \epsilon < \epsilon_0$, $p \in \Sigma$, and $q \in C_f(\Sigma, \epsilon)$, $f(\xi_\epsilon(p)) = \epsilon$ and $f(\xi_{-\epsilon}(q)) = 0$. This means that $\xi_{\epsilon|\Sigma} : \Sigma \rightarrow C_f(\Sigma, \epsilon)$ is a diffeomorphism with inverse $\xi_{-\epsilon}$. Let $\vartheta(\epsilon)$ and $\tilde{\vartheta}(\epsilon)$ be the coefficients of dilatation of ξ_ϵ and $\xi_{\epsilon|\Sigma}$. From $\xi_0(q) = q$, $\forall q$, we easily see that both $\vartheta(\epsilon)(q) \rightarrow 1$, $\tilde{\vartheta}(\epsilon)(p) \rightarrow 1$, when $\epsilon \rightarrow 0$. Moreover if $\eta(\Phi)$ can be defined as an L^1 -form along Σ , then (5.68) = $-\int_\Sigma \eta(\Phi)$. Unfortunately the case Σ a hypersurface is the least interesting, for, non J -complex Cayley submanifolds of \mathbb{R}^8 cannot have \mathcal{C} as an analytic hypersurface [12].

A key example is of $f = \sigma$ the intrinsic distance function to a smooth submanifold Σ of dimension d , $\sigma(q) = d(q, \Sigma) = \inf_{p \in \Sigma} d(q, p)$. In this case, ∇f is not well defined at each point $p \in \Sigma$, but $\|\nabla f\| = 1$, everywhere. In fact ∇f it is multivalued, with sublimits all unit normal vectors to Σ in M . Nevertheless the flow can be smoothly extended to Σ , in all directions of $T_p \Sigma^\perp$. We explain as follows. Let $N\Sigma$ denote the total space of the normal bundle of Σ in TM and $N^1\Sigma$ the spherical subbundle of the unit orthogonal vectors. For each $\epsilon > 0$ let

$$G_\epsilon = \{(p, w) \in N\Sigma : p \in \Sigma, w \in T_p \Sigma^\perp, \|w\| < \epsilon\}, \quad C_\epsilon = \{(p, w) \in N\Sigma : p \in \Sigma, w \in T_p \Sigma^\perp, \|w\| = \epsilon\}.$$

For $0 < \epsilon \leq \epsilon_0$, with ϵ_0 sufficiently small, the restriction of the exponential map of M , $\exp : G_\epsilon \rightarrow M$, $\exp(p, w) = \exp_p(w)$ defines a diffeomorphism onto $V_\sigma(\Sigma, \epsilon)$ and $\exp(C_\epsilon) = C_\sigma(\Sigma, \epsilon)$ is its boundary. For each $w \in T_p \Sigma^\perp$, $\gamma_{(p, w)}(\epsilon) = \exp_p(\epsilon w)$ is the geodesic normal to Σ , starting at p with initial velocity $w \in T_p \Sigma^\perp$. Thus, $s(p, w) := \sigma(\exp(p, w)) = \|w\|$, is just the Euclidean norm in $T_p \Sigma^\perp$. Since $N\Sigma$ is the total space of a Riemannian vector bundle, then it has a natural Riemannian structure such that $\pi : N\Sigma \rightarrow \Sigma$ is a Riemannian submersion. The volume element $Vol_{N\Sigma}$ for such metric satisfies $Vol_{N\Sigma}(p, w) = Vol_\Sigma(p) \wedge ds(p, w)$ and $Vol_{C_\epsilon}(p, w) = Vol_\Sigma(p) \wedge Vol_{S(p, \epsilon)}(w)$, where $\epsilon = \|w\|$, and $S(p, \epsilon)$ is the sphere of $T_p \Sigma^\perp$ of radius ϵ . For each $u \in N^1 \Sigma_p$, $\vartheta_{(p, u)}(\epsilon) = \langle Vol_{N\Sigma}(p, \epsilon u), \exp^* Vol_M(p, \epsilon u) \rangle$ is the coefficient of dilatation that measures the volume distortion by \exp in the direction u . It satisfies $\vartheta_{(p, u)}(0) = 1$. We recall the following (see [11]): (1) $\nu(q) = \nabla \sigma(q)$ is the unit outward of $C(\Sigma, \epsilon)$, (2) $\nu(\gamma_{(p, u)}(\epsilon)) = \gamma'_{(p, u)}(\epsilon)$, (3) $ds \wedge *ds$ and $d\sigma \wedge *d\sigma$ are the volume elements of $N\Sigma$ and M respectively. (4) $*ds$ and $*d\sigma$ are the volume elements of each hypersurface C_ϵ of $N\Sigma$ and $C(\Sigma, \epsilon)$ of M respectively, and $\exp^*(d\sigma)(p, w) = \vartheta_{(p, \frac{w}{\epsilon})}(\epsilon)(*ds)(p, w)$, where $\epsilon = \|w\|$. The set of sublimits of $\nabla \sigma$ at a point $p \in \Sigma$ is the entire sphere $S(p, 1)$ of $T_p \Sigma^\perp$ and by (1) and (2) for each $u \in N^1 \Sigma$, $\gamma_{(p, u)}(t)$ is an integral curve of $\nabla \sigma$ smoothly extended at $t = 0$ by p and initial velocity u . The map $\xi : N\Sigma \rightarrow V$, $\xi(p, tu) = \gamma_{(p, u)}(t)$, can be seen as the flow of $\nabla \sigma$, a vector field multivalued at Σ .

This example motivates the following. We consider functions $f : V \rightarrow \mathbb{R}_0^+$ satisfying the following conditions (E-1) and (E-2), that generalizes the case of $f = \sigma$. Let $X_f = \frac{\nabla f}{\|\nabla f\|^2}$ and Σ a smooth closed submanifold of dimension d .

(E-1) f is a nonnegative continuous function with zero set Σ , smooth on $V \sim \Sigma$ and with $\|\nabla f\|$ defined $\forall p \in \Sigma$, giving a positive function of class C^μ on M , and such that $\{u \in T_p M : u \text{ is a sublimit of } \frac{\nabla f}{\|\nabla f\|} \text{ at } p\} = N^1 \Sigma_p$.

Set for $p \in \Sigma$, $c(p) = \lim_{q \rightarrow p} \|\nabla f(q)\| > 0$. We are considering the sublimits defined through orthogonal curves to Σ , $\rho : [0, 1] \rightarrow M$, such that $\rho([0, 1]) \subset V \sim \Sigma$ and $\rho(0) \in \Sigma$, and exist $u = \lim_{t \rightarrow 0} \frac{\nabla f}{\|\nabla f\|}(\rho(t)) \in N^1 \Sigma_p$. The set $E\Sigma_p = \{\frac{u}{c(p)} : u \in N^1 \Sigma_p\}$ is just the set of sublimits of X_f at p . An integral curve $\gamma :]0, b[\rightarrow V \sim \Sigma$ of X_f has an end point at 0 converging to Σ with initial velocity $\frac{u}{c(p)}$ where $u \in N\Sigma'_p$, if $\exists \lim_{t \rightarrow 0^+} \gamma(t) = p$ and $\exists \lim_{t \rightarrow 0^+} \gamma'(t) = \frac{u}{c(p)}$.

(E-2) X_f has an extensible flow to Σ , i.e. $\forall (p, u) \in N^1 \Sigma$ there exist a $C^{\mu+1}$ curve $\gamma_{(p, u)}(t)$, defined $\forall t \in [0, t_0]$, smooth for $t > 0$, that satisfies:

- (a) for $t > 0$, $\gamma_{(p, u)}(t)$ is an integral curve of X_f on $V \sim \Sigma$, and $\gamma_{(p, u)}(0) = p$, $\gamma'_{(p, u)}(0) = \frac{u}{c(p)}$.
- (b) The flow at Σ , $\xi : N\Sigma \rightarrow V$, defined for (p, w) with $\|w\| < t_0$, by $\xi(p, 0) = p$, $\xi(p, w) = \gamma_{(p, u)}(\epsilon)$, where $u = \frac{w}{\|w\|}$ and $\epsilon = \|w\|$, is a diffeomorphism of class $C^{\mu+1}$.

So we have for $0 \leq \epsilon < t_0$, $\xi_\epsilon : C_\epsilon \rightarrow C_f(\Sigma, \epsilon)$. The coefficient of dilatation of ξ at $(p, \epsilon u)$, $\vartheta_{(p, u)}(\epsilon) = \langle Vol_{N\Sigma}(p, \epsilon u), (\xi^* Vol_M)(p, \epsilon u) \rangle$, satisfies $\vartheta_{(p, u)}(0) = \frac{1}{c(p)^{4-d}}$, as we will see below. The volume element of $C_f(\Sigma, \epsilon)$ is $\frac{*df}{\|\nabla f\|}$, for $\frac{\nabla f}{\|\nabla f\|}$ is the outward unit. Since $\frac{df}{\|\nabla f\|} \wedge \frac{*df}{\|\nabla f\|}$ is the volume element of M and $\xi^*(\frac{df}{\|\nabla f\|})(p, \epsilon u)(0, u) = \frac{df(\gamma'_{(p, u)}(\epsilon))}{\|\nabla f(\xi(p, \epsilon u))\|} = \frac{1}{\|\nabla f(\xi(p, \epsilon u))\|}$, then on $T_{(p, \epsilon u)} C_\epsilon$ we have $\frac{1}{\|\nabla f(p, \epsilon u)\|} \xi^*(\frac{*df}{\|\nabla f\|})(p, \epsilon u) = \vartheta_{(p, u)}(\epsilon) Vol_{C_\epsilon} = \vartheta_{(p, u)}(\epsilon) *ds$. Therefore, $\tilde{\vartheta}_{(p, u)}(\epsilon) :=$

$\|\nabla f(\xi(p, \epsilon u))\| \vartheta_{(p,u)}(\epsilon)$ is the coefficient of dilatation of ξ restricted to C_ϵ , and $\lim_{\epsilon \rightarrow 0} \tilde{\vartheta}_{(p,u)}(\epsilon) = c(p)^{d-3}$.

From (E-2) we have a coordinate system of class $C^{\mu+1}$ of Farmi-type. Let O be an open set of Σ where a coordinate system exist and a d.o.n. frame E_{d+1}, \dots, E_4 of $N\Sigma$. Then for each $w \in N\Sigma_p$, $w = \sum_{d+1 \leq i \leq 4} t_i E_i(p)$. Define on V' the image by ξ of the restriction to O ,

$$\begin{aligned} x : V' &\rightarrow N\Sigma &\rightarrow \mathbb{R}^4 \\ \xi(p, w) &\rightarrow (p, \sum_{d+1 \leq i \leq 4} t_i E_i(p)) &\rightarrow (y(p), t_{d+1}, \dots, t_4) \end{aligned}$$

Thus, for $q = \xi(p, \epsilon u) \in C_f(\Sigma, \epsilon)$, $\|u(p)\| = 1$, $u(p) = \sum_{1+d \leq i \leq 4} u_i(p) E_i(p)$

$$\partial_i f(q) = \begin{cases} 0 & \forall i \leq d \\ u_i(p) = \frac{x_i(q)}{f(q)} & \forall i \geq d+1 \end{cases} \quad (5.69)$$

$$\text{Hess } f(q)(\partial_i, \partial_j) = \begin{cases} -\sum_{s \geq d+1} \Gamma_{ij}^s(q) u_s(p) & \text{if } i \leq d \text{ or } j \leq d \\ \frac{1}{\epsilon}(\delta_{ij} - u_i(p)u_j(p)) - \sum_{s \geq d+1} \Gamma_{ij}^s(q) u_s(p) & \text{if } i, j \geq 1+d \end{cases} \quad (5.70)$$

In fact for $s \leq \mu$, it is defined a tensor $T^s \in C^{\mu-s}(\pi^{-1} \otimes^s TM^*)$, where $\pi : N^1\Sigma \rightarrow O \subset \Sigma$, and s.t. $\exists \lim_{\epsilon \rightarrow 0} \epsilon^{s-1} \nabla_{\partial_{i_s}, \dots, \partial_{i_1}}^s f(\xi(p, \epsilon u)) =: T^s(p, u)(\partial_{i_1}(p), \dots, \partial_{i_s}(p))$ where $\nabla_X f = df(X)$,

$$\nabla_{X_k, \dots, X_1}^k f = \nabla_{X_k}(\nabla_{X_{k-1}, \dots, X_1}^{k-1} f) - \sum_{k-1 \geq i \geq 1} \nabla_{X_{k-1}, \dots, \nabla_{X_k} X_i, \dots, X_1}^{k-1} f. \quad (5.71)$$

Recall that $T_{(p,0)}N\Sigma = T_p\Sigma \times T_p\Sigma^\perp$, and from $\xi(p, 0) = \gamma_{(p,u)}(0) = p$ we get $\forall X \in T_p\Sigma$, $d\xi_{(p,0)}(X, 0) = X$. Now, if $0 \neq h \in T_p\Sigma^\perp$, the curve $\tau(s) = \xi(p, sh) = \gamma_{(p, \frac{h}{\|h\|})}(\|h\|s)$ satisfies $\tau'(0) = d\xi_{(p,0)}(0, h) = \|h\| \gamma'_{(p, \frac{h}{\|h\|})}(0) = \frac{h}{c(p)}$. Thus, $d\xi_{(p,0)}(X, h) = X + \frac{h}{c(p)}$, and so $\vartheta_{(p,u)}(0) = (c(p))^{d-4}$. So we conclude:

Proposition 5.8. *If a continuous function f satisfies (E-1) and (E-2) then for each $p \in \Sigma$ there exist a $C^{\mu+1}$ coordinate chart x of M , adapted to Σ , and such that $f^2 = x_{d+1}^2 + \dots + x_4^2$ and for $i \geq d+1$, $E_i(p) = c(p) \frac{\partial}{\partial x_i}(p)$ is an o.n. basis of $T_p\Sigma^\perp$.*

From now on we assume $\sin \theta = f^r$, with $\Sigma = f^{-1} = \mathcal{C}$ and f satisfying conditions (E-1) and (E-2), with $\mu \geq r+1$. Set $\tilde{\Phi} = \frac{\Phi}{\|\Phi\|} = \frac{\Phi}{\sin \theta}$. Then $\tilde{\Phi}$ is an isometry and $\tilde{\Phi}^{-1} = \tilde{\Xi} := \frac{\Xi}{\|\Xi\|}$.

Using a $C^{\mu+1}$ coordinate chart with $\mu \geq r$, Φ has a zero of order r at 0 iff $\frac{\Phi(x)}{\|x\|^{r-1}} \rightarrow 0$ and $\frac{\Phi(x)}{\|x\|^r}$ does not converge to 0 when $x \rightarrow 0$, in other words, Φ (or equivalently $\|\Phi\|$) is an $O(\|x\|^r)$. This is equivalent to $D^s \Phi(0) = 0 \forall s \leq r-1$ and $D^r \Phi(0) \neq 0$. Thus, at all points $p \in \Sigma$, r is the order of the zero of Φ at p . Fix $p \in \Sigma$ and y a coordinate system of Σ with $y(p) = 0$, and consider x the corresponding Farmi coordinate system. Let $V'' = x(V')$ open set of \mathbb{R}^4 and $dx^{-1} : \mathbb{R}_{V''}^4 \rightarrow TV''$, and an isomorphism $\tau : NV'' \rightarrow \mathbb{R}_{V''}^4$. Then $P = \tau \circ \Phi_{x^{-1}} \circ d(x^{-1}) : V'' \rightarrow L(\mathbb{R}^4; \mathbb{R}^4)$ has at 0 a zero of order r . Thus, for v sufficiently close to 0

$$P(v) = \frac{1}{r!} D^r P(0)(v)^r + \int_0^1 \frac{(1-t)^r}{r!} D^{r+1} P(tv)(v)^{r+1} dt \quad (5.72)$$

where $D^r P(0)(v)^r = \tau_p \circ \nabla_{(d(x^{-1})(0)(v))^r}^r \Phi(p) \circ d(x^{-1})(0)$. This term does not vanish for some v . If we take $u = \sum_{i \geq d+1} t_i E_i \in N^1 \Sigma_p$, ϵ small enough, and $v = (0, \epsilon t_{d+1}, \dots, \epsilon t_4)$, then $x^{-1}(sv) = \gamma_{(p,u)}(s\epsilon)$ and so $d(x^{-1})(tv)(v) = \epsilon \gamma'_{(p,u)}(t\epsilon)$, $d(x^{-1})(0)(v) = \epsilon c(p)^{-1}u$. Therefore

$$\begin{aligned} \Phi(\xi(p, \epsilon u)) &= \epsilon^r (A(\xi(p, \epsilon u)) + \epsilon Q(\xi(p, \epsilon u))) \\ A(\xi(p, \epsilon u)) &= \frac{1}{r!c(p)^r} \tau_{\xi(p, \epsilon u)}^{-1} \circ \tau_p \nabla_{u^r}^r \Phi(p) \circ dx^{-1}(0) \circ dx(\xi(p, \epsilon u)) \quad \text{of class } C^\mu \end{aligned} \quad (5.73)$$

for some Q of class $C^{\mu-r}$. Let $\nabla_{X_k, \dots, X_1}^k \Phi(p)$ be defined as (5.71).

Lemma 5.7. $\forall X_i \in T_p \Sigma$, $\nabla_{X_r, \dots, X_1}^r \Phi(p) = 0$.

Proof. If X is a vector of $T_p \Sigma$, we can assume $X(p) = \gamma'(0)$ for some curve $\gamma(t)$ on Σ . Then $\nabla_X \Phi(p) = \nabla_{\frac{d}{dt}} \gamma^{-1} \Phi(0)$. But $\gamma^{-1} \Phi$ is constantly equal to 0. So if X is a vector field of Σ , $\nabla_X \Phi$ vanish along Σ . Moreover, $\nabla_{X_s, \dots, X_1}^s \Phi(p) = 0$ for any $s \leq r-1$ and $X_i \in T_p M$. If $r \geq 2$ and $X, Y \in T_p \Sigma$, we extend X to a vector field along Σ , and so, $\nabla_{Y, X}^2 \Phi(p) = \nabla_Y (\nabla_X \Phi)(p) = 0$, that is $\nabla_{Y, X}^2 \Phi$ vanish along Σ . This implies that if $r \geq 3$, for X, Y, Z vector fields of Σ , $\nabla_{Z, Y, X}^3 \Phi(p) = \nabla_Z (\nabla_{Y, X}^2 \Phi)(p) = 0$. The same for any r . \square

A tensor ς in $C^\infty(N\Sigma^* \otimes (T^* \Sigma \otimes N\Sigma))$ is defined by: if $u \in T_p \Sigma^\perp$, $X \in T_p \Sigma$ then $\varsigma(u)(X) = (\nabla_u \tilde{X})^\perp$, where \tilde{X} is any vector field of M with $\tilde{X}_p = X$.

Proposition 5.9. $\forall (p, u) \in N^1 \Sigma$, $\exists \lim_{\epsilon \rightarrow 0} \tilde{\Phi}(\xi(p, \epsilon u)) = \frac{1}{r!c(p)^r} \nabla_{u^r}^r \Phi(p) =: \tilde{\Upsilon}(p, u)$, is an isometry. Moreover $\forall X \in T_p \Sigma$, $Y_i \in T_p M$, $\nabla_{Y_{r-1}, \dots, Y_1, X}^r \Phi(p) = \nabla_{Y_{r-1}, \dots, Y_s, X, Y_{s+1}, \dots, Y_1}^r \Phi(p) = 0$, and $\nabla_{u^r}^r (\nabla_X \Phi) = \nabla_{u^r, X}^{r+1} \Phi(p) + r \nabla_{u^{r-1}, \varsigma(u)(X)}^r \Phi$.

Proof. $\frac{\Phi(\xi(p, \epsilon u))}{\epsilon^r} = \tilde{\Phi}(\xi(p, \epsilon u))$ is an isometry. By (5.73), making $\epsilon \rightarrow 0$ we conclude that $\forall u \in N^1 \Sigma_p$, $\tilde{\Upsilon}(p, u) = \frac{1}{r!c(p)^r} \nabla_{u^r}^r \Phi(p)$ is an isometry. Extend X to a local section of $T\Sigma$, and then extend X and Y_i to local sections of TM . Recall that $\nabla^s \Phi$ vanish along Σ , $\forall s \leq r-1$. Then $\nabla_X (\nabla_{Y_{r-1}, \dots, Y_1}^{(r-1)} \Phi)(p) = 0$, and

$$\begin{aligned} 0 &= \nabla_X (\nabla_{Y_{r-1}, \dots, Y_1}^{(r-1)} \Phi)(p) = \nabla_X (\nabla_{Y_{r-1}} (\nabla_{Y_{r-2}, \dots, Y_1}^{(r-2)} \Phi))(p) \\ &= \nabla_{Y_{r-1}} (\nabla_X (\nabla_{Y_{r-2}, \dots, Y_1}^{(r-2)} \Phi))(p) + \bar{R}(Y_{r-1}, X) (\nabla_{Y_{r-2}, \dots, Y_1}^{(r-2)} \Phi)(p) \\ &= \nabla_{Y_{r-1}} (\nabla_X (\nabla_{Y_{r-2}, \dots, Y_1}^{(r-2)} \Phi))(p) = \nabla_{Y_{r-1}} (\nabla_X (\nabla_{Y_{r-2}} (\nabla_{Y_{r-3}, \dots, Y_1}^{(r-3)} \Phi)))(p) \\ &= \nabla_{Y_{r-1}} (\nabla_{Y_{r-2}} (\nabla_X (\nabla_{Y_{r-3}, \dots, Y_1}^{(r-3)} \Phi)) + \bar{R}(Y_{r-2}, X) (\nabla_{Y_{r-3}, \dots, Y_1}^{(r-3)} \Phi))(p) \\ &= \nabla_{Y_{r-1}} (\nabla_{Y_{r-2}} (\nabla_X (\nabla_{Y_{r-3}, \dots, Y_1}^{(r-3)} \Phi)))(p) + \bar{R}(Y_{r-2}, X) (\nabla_{Y_{r-1}} (\nabla_{Y_{r-3}, \dots, Y_1}^{(r-3)} \Phi)(p)) \\ &= \nabla_{Y_{r-1}} (\nabla_{Y_{r-2}} (\nabla_X (\nabla_{Y_{r-3}, \dots, Y_1}^{(r-3)} \Phi))(p) \end{aligned}$$

and successively, $0 = \nabla_X (\nabla_{Y_{r-1}, \dots, Y_1}^{(r-1)} \Phi)(p) = \nabla_{X, Y_{r-1}, \dots, Y_1}^r \Phi(p) = \nabla_{Y_{r-1}, \dots, Y_s, X, Y_{s+1}, \dots, Y_1}^{(r-1)} \Phi(p) = \nabla_{Y_{r-1}, \dots, Y_1}^r (\nabla_X \Phi)(p) = \nabla_{Y_{r-1}, \dots, Y_1, X}^r \Phi(p)$. Thus, if $r = 2$ then $\nabla_{u^2}^2 (\nabla_X \Phi)(p) = \nabla_{u^2, X}^3 \Phi(p) + 2 \nabla_{u, \nabla_u X}^2 \Phi(p)$, and $\nabla_{u, \nabla_u X}^2 \Phi(p) = \nabla_{u, (\nabla_u X)^\perp}^2 \Phi(p) = \nabla_{u, \varsigma(u)(X)}^2 \Phi(p)$. The proof for $r \geq 3$ is similar, slightly more complicate. \square

Proposition 5.10. $\forall X$ vector field on M and $\forall (p, u) \in N^1\Sigma$, set $X^{\perp u} = X_p - g(X_p, u)u$. Then $\exists \lim_{\epsilon \rightarrow 0} \epsilon \nabla_X \tilde{\Phi}(\xi(p, \epsilon u)) = \frac{1}{(r-1)!c(p)^{r-1}} \nabla_{u^{r-1}, X^{\perp u}}^r \Phi(p) =: \tilde{\Psi}(p, u)(X^{\perp u})$. If $d \geq 1$ and $X_p \in T_p\Sigma$ and for q near p , $X(q) \perp \nabla f(q)$ then $\exists \lim_{\epsilon \rightarrow 0} \nabla_X \tilde{\Phi}(\xi(p, \epsilon u)) = \frac{1}{r!c(p)^r} (\nabla_{u^r, X}^{r+1} \Phi(p) + r \nabla_{u^{r-1}, \varsigma(u)(X)}^r \Phi(p)) =: \tilde{G}(p, u)(X_p)$.

Proof. There exist some \tilde{Q} of class $C^{\mu-r-1}$, s.t.

$$\begin{aligned} \nabla_X \Phi(\xi(p, \epsilon u)) &= \epsilon^{r-1} (B(X, \xi(p, \epsilon u)) + \epsilon \tilde{Q}(\xi(p, \epsilon u))) \\ B(X, \xi(p, \epsilon u)) &= \frac{1}{(r-1)!c(p)^{(r-1)}} \tau_{\xi(p, \epsilon u)}^{-1} \circ \tau_p \nabla_{u^{(r-1)}, X}^r \Phi(p) \circ dx^{-1}(0) \circ dx(\xi(p, \epsilon u)). \end{aligned} \quad (5.74)$$

Thus

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon \nabla_X \tilde{\Phi}(\xi(p, \epsilon u)) &= \lim_{\epsilon \rightarrow 0} \epsilon^{-r+1} \nabla_X \Phi(\xi(p, \epsilon u)) - r g(\nabla f(\xi(p, \epsilon u)), X) \tilde{\Phi}(\xi(p, \epsilon u)) \\ &= \lim_{\epsilon \rightarrow 0} B(X, \xi(p, \epsilon u)) + \epsilon \tilde{Q}(\xi(p, \epsilon u)) - r g(\nabla f(\xi(p, \epsilon u)), X) \tilde{\Phi}(\xi(p, \epsilon u)) \\ &= \frac{1}{(r-1)!c(p)^{(r-1)}} \nabla_{u^{(r-1)}, X}^r \Phi(p) - \frac{1}{(r-1)!c(p)^{r-1}} g(u, X) \nabla_{u^r}^r \Phi(p) \\ &= \frac{1}{(r-1)!c(p)^{(r-1)}} \nabla_{u^{(r-1)}, X^{\perp u}}^r \Phi(p). \end{aligned}$$

If $X_p \in T_p\Sigma$ then by Prop. 5.9, $B(X, \xi(p, \epsilon u)) = 0$, and $\nabla_X \Phi(\xi(p, \epsilon u)) = \epsilon^r \tilde{Q}(\xi(p, \epsilon u))$, and $\lim_{\epsilon \rightarrow 0} \tilde{Q}(\xi(p, \epsilon u)) = \frac{1}{r!c(p)^r} \nabla_{u^r}^r (\nabla_X \Phi)(p)$. Set $X = \partial_i$ where $i \leq d$. By (5.69) those vector fields span exactly the ones that are orthogonal to ∇f at $\xi(p, \epsilon u)$. Note that $X_p = \frac{\partial}{\partial y_i}(p)$. Hence, by Prop.5.9, $\lim_{\epsilon \rightarrow 0} \nabla_X \tilde{\Phi}(\xi(p, \epsilon u)) = \lim_{\epsilon \rightarrow 0} \epsilon^{-r} \nabla_X \Phi(\xi(p, \epsilon u)) = \frac{1}{r!c(p)^r} \nabla_{u^r}^r (\nabla_X \Phi)(p) = \tilde{G}(p, u)(X_p)$. \square

Therefore if $X_p \perp u$,

$$\lim_{\epsilon \rightarrow 0} \epsilon \tilde{\Phi}^{-1} \nabla_X \tilde{\Phi}(\xi(p, \epsilon u)) = \tilde{\Upsilon}(p, u)^{-1} \tilde{\Psi}(p, u)(X_p) = rc(p) \left(\nabla_{u^r}^r \Phi(p) \right)^{-1} \circ \left(\nabla_{u^{(r-1)}, X}^r \Phi(p) \right), \quad (5.75)$$

and if $X \in T_p\Sigma$ and $X(q) \perp \nabla f(q)$ for q near p ,

$$\lim_{\epsilon \rightarrow 0} \tilde{\Phi}^{-1} \nabla_X \tilde{\Phi}(\xi(p, \epsilon u)) = \tilde{\Upsilon}(p, u)^{-1} \tilde{G}(p, u)(X_p) = \left(\nabla_{u^r}^r \Phi(p) \right)^{-1} \circ \left(\nabla_{u^r, X}^{r+1} \Phi(p) + r \nabla_{u^{r-1}, \varsigma(u)(X)}^r \Phi(p) \right). \quad (5.76)$$

We can write $\Phi = \sin \theta \tilde{\Phi}$ where $\tilde{\Phi} : TM \rightarrow NM$ is an isometry, away from Σ . More generally,

Lemma 5.8. If V is an open set containing Σ , and on $V \sim \Sigma$ $\Phi = \zeta^r \tilde{\Phi}$, where $\tilde{\Phi}$ is a section of $TM^* \otimes NM$ defined on $V \sim \Sigma$ and $\zeta : V \sim \Sigma \rightarrow \mathbb{R}_0^+$ some function, then where ζ and $\tilde{\Phi}$ are differentiable and do not vanish, $\eta(\Phi) = \eta(\tilde{\Phi})$

Proof. $\tilde{\Phi} : TM \rightarrow NM$ is a conformal morphism. So (5.13) holds for $\tilde{\Phi}$ with $\tilde{\varphi} = \log(\frac{\|\tilde{\Phi}\|^2}{4})$. and

$$\begin{aligned} \Phi^{-1} \nabla_X \Phi &= rd \log \zeta(X) Id_{TM} + \tilde{\Phi}^{-1} \nabla_X \tilde{\Phi} \\ \Phi(R^\perp) &= \tilde{\Phi}(R^\perp), \quad b(\Phi(R^\perp)) = b(\tilde{\Phi}(R^\perp)) = \tilde{\Phi}^{-1} d^2 \tilde{\Phi} \\ d^2 \Phi &= \zeta^r d^2 \tilde{\Phi}, \quad d(\Phi^{-1} d\Phi) = d(\tilde{\Phi}^{-1} d\tilde{\Phi}) \end{aligned}$$

The last two equalities are proved using the symmetry of $Hess \log \zeta$. Now

$$\begin{aligned}\langle Id_{TM}, \tilde{\Phi}(R^\perp)(Y, Z) \rangle &= \sum_i \zeta^{-r} h^{-1} g(\tilde{\Phi}(e_i), R^\perp(Y, Z)(\tilde{\Phi}(e_i))) = 0, \\ \langle Id_{TM}, R^M(Y, Z) \rangle &= \sum_i g(e_i, R^M(Y, Z)(e_i)) = 0, \\ [(rd \log \zeta(\cdot) Id_{TM} + \tilde{\Phi}^{-1} \nabla \tilde{\Phi}), (rd \log \zeta(\cdot) Id + \tilde{\Phi}^{-1} \nabla \tilde{\Phi})] &= [\tilde{\Phi}^{-1} \nabla \tilde{\Phi}, \tilde{\Phi}^{-1} \nabla \tilde{\Phi}].\end{aligned}$$

Thus, $\eta(\Phi) = \eta(\tilde{\Phi}) - \frac{r}{12\epsilon} \langle (d\zeta(\cdot) Id_{TM}) \wedge [\tilde{\Phi}^{-1} \nabla \tilde{\Phi}, \tilde{\Phi}^{-1} \nabla \tilde{\Phi}] \rangle$. Moreover

$$\begin{aligned}\langle (d\zeta(\cdot) Id_{TM}) \wedge [\tilde{\Phi}^{-1} \nabla \tilde{\Phi}, \tilde{\Phi}^{-1} \nabla \tilde{\Phi}] \rangle(X, Y, Z) &= \\ &= \bigoplus_{X, Y, Z} \zeta_X \langle Id, \tilde{\Phi}^{-1} \nabla_Y \tilde{\Phi} \circ \tilde{\Phi}^{-1} \nabla_Z \tilde{\Phi} - \tilde{\Phi}^{-1} \nabla_Z \tilde{\Phi} \circ \tilde{\Phi}^{-1} \nabla_Y \tilde{\Phi} \rangle \\ &= \bigoplus_{X, Y, Z} \sum_i \zeta_X g(e_i, \tilde{\Phi}^{-1} \nabla_Y \tilde{\Phi}(\tilde{\Phi}^{-1} \nabla_Z \tilde{\Phi}(e_i))) - \zeta_X g(e_i, \tilde{\Phi}^{-1} \nabla_Z \tilde{\Phi}(\tilde{\Phi}^{-1} \nabla_Y \tilde{\Phi}(e_i))) \\ &= \bigoplus_{X, Y, Z} \sum_i -\zeta_X g(\tilde{\Phi}^{-1} \nabla_Y \tilde{\Phi}(e_i), \tilde{\Phi}^{-1} \nabla_Z \tilde{\Phi}(e_i)) + \zeta_X \tilde{\varphi}_Y g(e_i, \tilde{\Phi}^{-1} \nabla_Z \tilde{\Phi}(e_i)) \\ &\quad \bigoplus_{X, Y, Z} \sum_i + \zeta_X g(\tilde{\Phi}^{-1} \nabla_Z \tilde{\Phi}(e_i), \tilde{\Phi}^{-1} \nabla_Y \tilde{\Phi}(e_i)) - \zeta_X \tilde{\varphi}_Z g(e_i, \tilde{\Phi}^{-1} \nabla_Y \tilde{\Phi}(e_i)) \\ &= 2\zeta_X \tilde{\varphi}_Y \tilde{\varphi}_Z - 2\zeta_X \tilde{\varphi}_Z \tilde{\varphi}_Y = 0\end{aligned}$$

Thus, $\eta(\Phi) = \eta(\tilde{\Phi})$. □.

Proof of Corollary 1.1. For simplicity of notation we assume $\Sigma_i = \Sigma$. By (1.6) of Theorem 1.1 and by Lemma 5.8 we have

$$p_1(\wedge_-^2 NM)[M] - p_1(\wedge_-^2 TM)[M] = \int_M d\eta(\Phi) = -\lim_{\epsilon \rightarrow 0} \int_{C_f(\Sigma, \epsilon)} \eta(\Phi) = -\lim_{\epsilon \rightarrow 0} \int_{C_f(\Sigma, \epsilon)} \eta(\tilde{\Phi})$$

Now

$$\begin{aligned}\int_{C_f(\Sigma, \epsilon)} \eta(\tilde{\Phi}) &= \int_{C(\Sigma, \epsilon)} \frac{1}{\|\nabla f\|^2} \langle \eta(\tilde{\Phi})(q), *df(q) \rangle *df(q) = \int_{C_\epsilon} \xi^* \left(\frac{1}{\|\nabla f\|^2} \langle \eta(\tilde{\Phi}), *df \rangle *df \right)(p, w) \\ &= \int_{C_\epsilon} \frac{1}{\|\nabla f(\xi(p, w))\|} \langle \eta(\tilde{\Phi})(\xi(p, w)), *df(\xi(p, w)) \rangle \vartheta_{(p, \frac{w}{\epsilon})}(\epsilon) (*ds)(p, w) \\ &= \int_\Sigma \left(\int_{S(p, \epsilon)} \frac{1}{\|\nabla f(\xi(p, w))\|} \langle \eta(\tilde{\Phi})(\xi(p, w)), *df(\xi(p, w)) \rangle \vartheta_{(p, \frac{w}{\epsilon})}(\epsilon) d_{S(p, \epsilon)}(w) \right) d_\Sigma(p) \\ &= \int_\Sigma \left(\int_{S(p, 1)} \frac{1}{\|\nabla f(\xi(p, \epsilon u))\|} \langle \eta(\tilde{\Phi})(\xi(p, \epsilon u)), *df(\xi(p, \epsilon u)) \rangle \vartheta_{(p, u)}(\epsilon) \epsilon^{3-d} d_{S(p, 1)}(u) \right) d_\Sigma(p)\end{aligned}$$

and $\frac{1}{\|\nabla f(\xi(p, \epsilon u))\|} \langle \eta(\tilde{\Phi})(\xi(p, \epsilon u)), *df(\xi(p, \epsilon u)) \rangle = \eta(\tilde{\Phi})(\xi(p, \epsilon u))(e_2, e_3, e_4)$, where $e_i \in T_{(\xi(p, \epsilon u))} C_f(\Sigma, \epsilon)$ is a d.o.n. frame. We take $e_1 := \frac{\nabla f(\xi(p, \epsilon u))}{\|\nabla f(\xi(p, \epsilon u))\|}$ the outward unit of $C_f(\Sigma, \epsilon)$ at $\xi(p, \epsilon u)$, and so $e_2 \wedge e_3 \wedge e_4 = \frac{* \nabla f(\xi(p, \epsilon u))}{\|\nabla f(\xi(p, \epsilon u))\|}$, giving e_i a d.o.n. basis of $T_{\xi(p, \epsilon u)} M$. Then

$$\int_{C_f(\Sigma, \epsilon)} \eta(\tilde{\Phi}) = \int_\Sigma \left(\int_{S(p, 1)} \eta(\tilde{\Phi}) \left(\frac{* \nabla f(\xi(p, \epsilon u))}{\|\nabla f(\xi(p, \epsilon u))\|} \right) \vartheta_{(p, u)}(\epsilon) \epsilon^{3-d} d_{S(p, 1)}(u) \right) d_\Sigma(p) \quad (5.77)$$

But $\frac{\nabla f}{\|\nabla f\|^2}(\xi(p, \epsilon u)) = \gamma'_{(p, u)}(\epsilon)$ where $\gamma_{(p, u)}(\epsilon) = \xi(p, \epsilon u)$. Then $\frac{* \nabla f(\xi(p, \epsilon u))}{\|\nabla f(\xi(p, \epsilon u))\|}$ converges to $*u$

when $\epsilon \rightarrow 0$. Recall that $\vartheta(p, u)(0) = \frac{1}{c(p)^{4-d}}$. Now, from Propositions 5.9 and 5.10,

$$\begin{aligned} \text{if } d = 2 \quad & \lim_{\epsilon \rightarrow 0} \epsilon \langle \tilde{\Phi}^{-1} \nabla \tilde{\Phi}(\xi(p, \epsilon u)) \wedge (\tilde{\Phi}(R^\perp) + R^M)(\xi(p, \epsilon u)) \rangle = \\ & = \langle \tilde{\Upsilon}(p, u)^{-1} \tilde{\Psi}(p, u) \wedge (\tilde{\Upsilon}(p, u)(R^\perp(p)) + R^M(p)) \rangle \\ \text{if } d = 0 \quad & \lim_{\epsilon \rightarrow 0} \epsilon^3 \langle \tilde{\Phi}^{-1} \nabla \tilde{\Phi}(\xi(p, \epsilon u)) \wedge [\tilde{\Phi}^{-1} \nabla \tilde{\Phi}(\xi(p, \epsilon u)), \tilde{\Phi}^{-1} \nabla \tilde{\Phi}(\xi(p, \epsilon u))] \rangle = \\ & = \langle \tilde{\Upsilon}(p, u)^{-1} \tilde{\Psi}(p, u) \wedge [\tilde{\Upsilon}(p, u)^{-1} \tilde{\Psi}(p, u), \tilde{\Upsilon}(p, u)^{-1} \tilde{\Psi}(p, u)] \rangle \end{aligned}$$

Now if $d \geq 1$, $\frac{* \nabla f(\xi(p, \epsilon u))}{\|\nabla f(\xi(p, \epsilon u))\|} = X_1 \wedge X_2 \wedge X_3$ with $X_i \perp \nabla f(\xi(p, \epsilon u))$. By (5.69), we may take X_1, \dots, X_d an o.n. basis of $\text{span}\{\partial_i, i \leq d\}$. Note that for $i \leq d$, $\lim_{\epsilon \rightarrow 0} \partial_i(\xi(p, \epsilon u)) = \frac{\partial}{\partial y_i}(p) \in T_p \Sigma$. This implies by Prop. 5.9, if $d = 3$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon^{3-d} \langle \tilde{\Phi}^{-1} \nabla \tilde{\Phi}(\xi(p, \epsilon u)) \wedge [\tilde{\Phi}^{-1} \nabla \tilde{\Phi}(\xi(p, \epsilon u)), \tilde{\Phi}^{-1} \nabla \tilde{\Phi}(\xi(p, \epsilon u))] \rangle & \left(\frac{* \nabla f(\xi(p, \epsilon u))}{\|\nabla f(\xi(p, \epsilon u))\|} \right) = \\ & = \langle \tilde{\Upsilon}(p, u)^{-1} \tilde{G}(p, u) \wedge [\tilde{\Upsilon}(p, u)^{-1} \tilde{G}(p, u), \tilde{\Upsilon}(p, u)^{-1} \tilde{G}(p, u)] \rangle \end{aligned}$$

If $d = 2$ or $d = 1$ we get similar expressions, noting for example that if $d = 1$, $\tilde{\Psi}(p, u)(X_1) = 0$ (see Prop. 5.10), and so

$$\langle \tilde{G}(p, u)(X_1), [\tilde{\Psi}(p, u)(X_2), \tilde{\Psi}(p, u)(X_3)] \rangle = \langle \tilde{G}(p, u) \wedge [\tilde{\Psi}(p, u), \tilde{\Psi}(p, u)] \rangle (X_1, X_2, X_3).$$

Using (5.73) and (5.74) we see that $\eta(\Phi) = \eta(\tilde{\Phi})$ is bounded by an L^1 form on $N^1 \Sigma$, and so we can apply the dominate convergence theorem to interchange \int with $\lim_{\epsilon \rightarrow 0}$ in (5.80), and the expression of Corollary 1.1 is proved \square .

Remark. For any symmetric tensor $S \in C^\infty(TM^* \otimes TM)$, $\langle S \wedge [\Phi^{-1} \nabla \Phi, \Phi^{-1} \nabla \Phi] \rangle = 0$. Thus the condition of $\langle \Phi^{-1} \nabla \Phi \wedge [\Phi^{-1} \nabla \Phi, \Phi^{-1} \nabla \Phi] \rangle = 0$ is a quite weaker condition than $\Phi^{-1} \nabla_X \Phi$ to be symmetric, for each vector field X . In this case, if $d_i \neq 2 \forall i$ then $p_1(\wedge_-^2 NM) = p_1(\wedge_-^2 TM)$.

6 J -Kähler submanifolds

Assume M is a Kähler submanifold of N . If E is a rank-4 Hermitian vector bundle over M with a complex structure J^E and a unitary connection ∇^E , the curvature is J^E -invariant. Let $B = (E_1, E_2 = J^E E_1, E_3, E_4 = J^E E_3)$ be a local o.n. frame of E , and Ξ_s^+ defined as J_s^B in (3.14), and Ξ_s^- defined in the same way, but replacing E_4 by $-E_4$. Let $c_1(E)$ and $c_2(E)$ be the first and the second Chern classes of E . Then

$$2\pi c_1(E) = R^E(\Xi_1^+) \quad R^E(\Xi_s^+) = 0 \text{ for } s = 2, 3. \quad (6.1)$$

$$c_2(E) = \mathcal{X}(E) \quad p_1(E) = -c_2(E^c) = -2c_2(E) + c_1(E)^2 \quad (6.2)$$

If $E = TM$ we denote J^E by J , E_i by e_i , and Ξ_s^\pm by Λ_s^\pm . Then $2\pi c_1(M)(X, Y) = \text{Ricci}^M(JX, Y) = R^M(\Lambda_1^+, X \wedge Y) = R^M(X \wedge Y, \Lambda_1^+)$. The first equation (6.1) implies that $c_1(E) = 0$ iff $\Lambda_1^+ E$ is flat. We also recall that (see e.g. [6])

$$\mathcal{X}(E) = \frac{1}{8\pi^2} (\|(R^E)_+^+\|^2 - \|(R^E)_+^-\|^2 - \|(R^E)_-^+\|^2 + \|(R^E)_-^-\|^2) \text{Vol}_M \quad (6.3)$$

$$p_1(E) = \frac{1}{4\pi^2} (\|(R^E)_+^+\|^2 + \|(R^E)_+^-\|^2 - \|(R^E)_-^+\|^2 - \|(R^E)_-^-\|^2) \text{Vol}_M \quad (6.4)$$

Since M is Kähler, NM is a Hermitian vector bundle and ∇^\perp is a unitary connection, and

$$F^*c_1(N) = c_1(M) + c_1(NM) \quad F^*c_2(N) = c_2(M) + c_1(M) \wedge c_1(NM) + c_2(NM) \quad (6.5)$$

$$p_1(M) + 2\mathcal{X}(M) = c_1(M)^2 \quad p_1(NM) + 2\mathcal{X}(NM) = c_1(NM)^2 \quad (6.6)$$

$$p_1(M) - 2\mathcal{X}(M) = c_1(M)^2 - 4c_2(M) \quad p_1(NM) - 2\mathcal{X}(NM) = c_1(NM)^2 - 4c_2(NM) \quad (6.7)$$

We define for $U, V \in NM_p$, $X, Y \in T_pM$

$$Ricci^\perp(U \wedge V) = R^\perp(\Lambda_1^+, U \wedge V), \quad Ricci_\perp(X \wedge Y) = R^\perp(X \wedge Y, \Xi_1^+) = 2\pi c_1(NM) \quad (6.8)$$

Lemma 6.1. *If M is a Kähler submanifold of N , then*

- (1) $(\|(R^\perp_+)^+\|^2 - \|(R^\perp_+)^-\|^2) = \frac{1}{2}Ricci^\perp \wedge Ricci^\perp(E_1, E_2, E_3, E_4)$
 $(\|(R^\perp_+)^+\|^2 - \|(R^\perp_+)^-\|^2) = \frac{1}{2}Ricci_\perp \wedge Ricci_\perp(e_1, e_2, e_3, e_4).$
- (2) $p_1(\Lambda_+^2 NM) = \frac{1}{4\pi^2}Ricci^\perp \wedge Ricci^\perp(E_1, E_2, E_3, E_4) Vol_M = \frac{1}{4\pi^2}Ricci_\perp \wedge Ricci_\perp.$
- (3) $\|(R^\perp_+)^+\| = \|(R^\perp_+)^-\|$ and $\|(R^{\perp^2 NM})^+\| = \|(R^{\perp^2 NM})^-\|.$

If we replace R^\perp by R^M the same equalities holds.

Proof. (1) Using (6.1)

$$\begin{aligned} 4(\|R^\perp_+^+\|^2 - \|R^\perp_+^-\|^2) &= \sum_{ts} (R_{\Xi_s^+}^\perp(\Lambda_t^+))^2 - (R_{\Xi_s^-}^\perp(\Lambda_t^+))^2 \\ &= (Ricci^\perp(\Xi_1^+))^2 - 2(Ricci^\perp(\Xi_1^-))^2 - 2(Ricci^\perp(\Xi_2^-))^2 - 2(Ricci^\perp(\Xi_3^-))^2 \\ &= (Ricci^\perp(E_1 \wedge JE_1) + Ricci^\perp(E_3, JE_3))^2 - (Ricci^\perp(E_1 \wedge JE_1) - Ricci^\perp(E_3 \wedge JE_3))^2 \\ &\quad - (Ricci^\perp(E_1 \wedge E_3) + Ricci^\perp(JE_1 \wedge JE_3))^2 - (Ricci^\perp(E_1 \wedge E_4) + Ricci^\perp(JE_1 \wedge JE_4))^2 \\ &= 4(Ricci^\perp(E_1 \wedge JE_1)Ricci^\perp(E_3 \wedge JE_3) - (Ricci^\perp(E_1 \wedge E_3))^2 - (Ricci^\perp(E_1 \wedge E_4))^2) \\ &= 2(Ricci^\perp(E_1 \wedge JE_1)Ricci^\perp(E_3 \wedge JE_3) - Ricci^\perp(E_1 \wedge E_3)Ricci^\perp(E_2 \wedge E_4) \\ &\quad + Ricci^\perp(E_1 \wedge E_4)Ricci^\perp(E_2 \wedge E_3)) \\ &= 4Ricci^\perp \wedge Ricci^\perp(E_1, E_2, E_3, E_4) \end{aligned}$$

Similar for the second equality. From (6.3), (6.4), (1) and (5.1), $p_1(\Lambda_+^2 NM) = \frac{1}{2\pi^2}(\|(R^\perp_+)^+\|^2 - \|(R^\perp_+)^-\|^2) Vol = \frac{1}{4\pi^2}Ricci^\perp \wedge Ricci^\perp(E_1, E_2, E_3, E_4) Vol$. But on the other hand by (6.7) and (6.8) $p_1(\Lambda_+^2 NM) = c_1^2(NM) = \frac{1}{4\pi^2}Ricci_\perp \wedge Ricci_\perp$. Thus $Ricci^\perp \wedge Ricci^\perp(E_1, E_2, E_3, E_4) = Ricci_\perp \wedge Ricci_\perp(e_1, e_2, e_3, e_4)$. So we have obtained (2). (3) Follows immediately from (1), (2) and that denoting by $\{1, 2, 3\} = \{\Xi_1^\pm, \Xi_2^\pm, \Xi_3^\pm\}$, one has $(R_{ab}^{\perp^2})^+ = \epsilon\sqrt{2}(R^E)_{\Xi_c^+}^+$, $(R_{ab}^{\perp^2})^- = \epsilon\sqrt{2}(R^E)_{\Xi_c^+}^-$, $(R_{ab}^{\perp^2})^+ = \epsilon\sqrt{2}(R^E)_{\Xi_c^-}^+$, $(R_{ab}^{\perp^2})^- = \epsilon\sqrt{2}(R^E)_{\Xi_c^-}^-$, where $\{a, b, c\}$ is a permutation of $\{1, 2, 3\}$ of signature ϵ . \square

Proposition 6.1. *If M is a complex submanifold of N , and $c_1(N) = 0$, then:*

- (1) $p_1(\Lambda_+^2 NM) = p_1(\Lambda_+^2 TM)$.
- (2) $p_1(\Lambda_-^2 NM) - p_1(\Lambda_-^2 TM) = 4(-F^*c_2(N) + 2c_2(M) - c_1(M)^2)$.
- (3) *If $c_1(M) = 0$, then both $\Lambda_+^2 TM$ and $\Lambda_+^2 NM$ are flat and both $\Lambda_-^2 TM$ and $\Lambda_-^2 NM$ are anti-self-dual. Moreover, $\mathcal{X}(M) \geq 0$ (resp. $\mathcal{X}(NM) \geq 0$) with equality to zero iff M (resp. NM) is flat. Furthermore, $F^*c_2(N)[M] \geq 0$ with equality to zero iff M and NM are flat.*

Proof. From (6.5), if $c_1(N) = 0$ then $c_1(M) = -c_1(NM)$ and so by (6.6) $p_1(\bigwedge_+^2 NM) = p_1(\bigwedge_+^2 TM)$. Now $p_1(\bigwedge_-^2 NM) - p_1(\bigwedge_-^2 TM) = -4\mathcal{X}(NM) + 4\mathcal{X}(M) = -4(c_2(NM) + 4c_2(M))$. (6.5) gives the first equality in (2). Now we assume $c_1(M) = 0$. Since $0 = F^*c_1(N) = c_1(M) + c_1(NM)$ along M , then $c_1(NM) = 0$, and both $\bigwedge_+^2 TM$ and $\bigwedge_+^2 NM$ are flat. By Lemma 6.1(3) $(R^{\bigwedge_-^2 NM})^+ = (R^{\bigwedge_-^2 TM})^+ = 0$, and so $\bigwedge_-^2 NM$ and $\bigwedge_-^2 TM$ are anti-selfdual. From (6.8) $Ricci_\perp = 0$, and since $(R^\perp)(\Xi_s^+) = 0 \quad \forall s = 1, 2, 3$, then $(R^\perp)_+^- = (R^\perp)_+^+ = 0$. From Lemma 6.1(3) we get $(R^\perp)_-^+ = 0$ as well. The same holds for R^M . The final statement is a consequence of the previous ones and that by (6.3)(6.4), with $E = TM$ or NM , $-p_1(E) = 2\mathcal{X}(E) = \frac{1}{4\pi^2} \int_M \|(R^E)_-^-\|^2$. \square

Remark. Part of Prop.6.1 (4) is a particular case of some results in [15] and in [4].

7 I -Kähler submanifolds

In subsection 3.2 we saw that if N is an HK manifold of complex dimension 4 and M is an I -Kähler submanifold, then the zero set Σ of $F^*\omega_J$ is the zero set of a globally defined I -holomorphic $(2,0)$ -form φ on M . Thus, Σ is a locally finite union of irreducible I -complex hypersurfaces Σ_i , and φ vanish to order a_i along Σ_i . Since $2\cos^2\theta = \|F^*\omega_J\|^2 = 2\|\varphi\|^2$, $\cos\theta$ vanish to homogeneous order a_i along Σ_i . $D = \sum_i a_i \Sigma_i$ is a divisor of φ , and for any closed 2-form ϕ of M

$$\int_M -\frac{i}{\pi} \partial\bar{\partial} \log \|\varphi\| \wedge \phi = \int_D \phi. \quad (7.1)$$

Proof of Proposition 1.1 By Theorem 3.1 we have $-i\partial\bar{\partial} \log \|\varphi\| \wedge \omega_I = Ricci(I(\cdot), \cdot) \wedge \omega_I = \frac{1}{2}s^M Vol_M$. If we take in (7.1) $\phi = \omega_I$, we get $\frac{1}{\pi}\kappa_2(M) = \int_D \omega_I = \sum_i a_i \int_{\Sigma_i} \omega_I$. \square

If I does not exist globally on M , we still can obtain a residue formula under some conditions. In [25] we introduced the notion of controlled zero set for a function on M with zero set a submanifold Σ . For each $(p, u) \in N^1\Sigma$ define $1 \leq \kappa(p, u) \leq +\infty$ the order of the zero of $\varphi_{(p,u)}(r) = \cos^2\theta(\exp_p(ru))$ at $r = 0$. We will say that $\cos^2\theta$ has a *controlled zero set* if there exist a nonnegative integrable function $f : N^1\Sigma \rightarrow [0, +\infty]$ and $r_0 > 0$ s.t. $\sup_{0 < r < r_0} |r \frac{d}{dr} \log(\varphi_{(p,u)}(r))| \leq f(p, u)$ a.e. $(p, u) \in N^1\Sigma$. For each $p \in \Sigma$, $S(p, 1)$ denotes the unit sphere of $T_p\Sigma^\perp$ and $\sigma_{d'}$ its volume. The function $\tilde{\kappa}(p) = \frac{1}{\sigma_{d'}} \int_{S(p,1)} \kappa(p, u) d_{S(p,1)} u$ is the *average order* of the zero p of $\cos^2\theta$, in the normal direction. Next proposition has a very similar proof to the one of Theorem 1.2 of [25], so we omit it.

Proposition 7.1. *Assume (N, J, g) is Ricci-flat KE and M is I -Kähler, closed, and Σ is a finite disjoint union of closed submanifolds Σ_i with dimension $d_i \leq 2$ and let $\bigcup_\gamma k_{i\gamma}$ be the range set of κ on $N^1\Sigma_i$ and define $N^1\Sigma_i^\gamma = \kappa^{-1}(k_{i\gamma})$. If κ is bounded a.e. and $\cos\theta$ has controlled zero set, then*

$$k_2(M) = - \sum_{i:d_i=2} \pi \int_{\Sigma_i} \tilde{\kappa}(p) Vol_{\Sigma_i} = -\frac{1}{2} \sum_{i:d_i=2} \sum_\gamma k_{i\gamma} Vol_{N^1\Sigma_i}(N^1\Sigma_i^\gamma).$$

As a consequence we have got a removable high rank singularity theorem:

Corollary 7.1. *In the conditions of Prop.7.1, $k_2(M) \leq 0$, with equality to zero iff $\Sigma_i = 0 \forall i : d_i = 2$.*

Now we prove

Proposition 7.2. *Let M be closed Cayley submanifold of a Ricci-flat Kähler-Einstein 4-fold (N, J, g) , that is not J -complex neither J -Lagrangian but it is I -Kähler on a open dense set U of $M \sim \mathcal{L}$. Then*

(1) $\forall p \geq 1, \int_M \cos^{2p} \theta s^M \text{Vol}_M \leq 0$. Consequently, $s^M \geq 0$ iff $s^M = 0$. If that is the case then $\cos \theta$ is constant.

(2) If M is immersed without J -Lagrangian points, then $2\kappa_2(M) = \int_M s^M \text{Vol}_M = 0$.

Proof. From (3.20) $\Delta \cos^{2p} \theta = p \cos^{2p} \theta s^M + 4p^2 \cos^{2p-2} \theta \|\nabla \cos \theta\|^2$. Integration and Stokes gives the inequalities in (1). If $s^M \geq 0$ and since the set of J -Lagrangian points has empty interior, (1) implies $s^M = 0$, and so $\Delta \cos^{2p} \theta \geq 0$. Thus, $\cos \theta$ is constant. Integration of (3.20) under the assumption of $\mathcal{L} = \emptyset$ proves (2). \square

Corollary 7.2. *If M is a closed I -complex 4-submanifold of an HK manifold (N, I, J, K, g) of real dimension 8, and if $s^M > 0$, then M is a totally complex submanifold.*

Proof. Quaternionic submanifolds are HK, and so Ricci-flat, what is not possible. If we assume M is not totally complex, by Proposition 7.1 s^M should vanish. \square

Proposition 7.3. *If M is a closed I -complex 4-submanifold of an HK manifold (N, I, J, K, g) of real dimension 8, at quaternionic points $s^M \leq 0$.*

Proof. Quaternionic points are maximum points of $\cos \theta$. Thus, by (3.20) $s^M \leq 0$. \square

Proposition 7.4. *Let M be a Cayley submanifold of a Ricci flat KE 8-manifold (N, J, g) , that is neither J -complex nor J -Lagrangian and it is a I -Kähler on a open set O of M . If $\cos \theta$ is constant on O then $(M, I, J_\omega, IJ_\omega)$ is HK on O .*

Proof. Since $\cos \theta$ is constant on O , by Prop.3.6, J_ω is Kähler on O , and so, M is HK on O . \square

The following proposition was already announced in [2] and can be also seen as a corollary of the above propositions:

Theorem 7.1. ([2]) *If M is a closed I -complex 4-submanifold of an HK manifold (N, I, J, K, g) of real dimension 8, and M is neither a quaternionic nor a totally complex submanifold, then the following assertions are equivalent to each other:*

- (a) $s^M = 0$
- (b) $\cos \theta_J$ is constant
- (c) $(M, I, J_{\omega_J}, J_{\omega_K}, g)$ is HK
- (d) the quaternionic angle of M is constant.

Remark. (b) implies (a) and (c), and (b) \iff (c) \iff (d) do not need compactness of M . The proof (d) \iff (b) is shown in [2]. In [1] we can find related results.

8 Cayley submanifolds of \mathbb{R}^8

8.1 Complex Cayley graphs

We consider $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$, with the euclidean metric g_0 (in \mathbb{R}^4 , and so in \mathbb{R}^8). The set of g -orthogonal complex structures of \mathbb{R}^8 has two connected components. Let us fix J_0 given by $J_0(X, Y) = (-Y, X)$, and denote by ω_0 the Kähler form. If $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is a smooth map, the graph of f is the map $\Gamma_f : \mathbb{R}^4 \rightarrow \mathbb{R}^8$, $\Gamma_f(x) = (x, f(x))$. In [7] we compute the Kähler angles of Γ_f with respect to J_0 . Let g_M be the the graph metric on \mathbb{R}^4 , $g_M = (\Gamma_f)^* g_0$. Note that $\Gamma_f^* \omega_0(X, Y) = g_0(-df(X) + df^t(X), Y)$, where g_0 is w.r.t. \mathbb{R}^4 . Using the musical isomorphism w.r.t. the Euclidean metric g_0 on \mathbb{R}^4 we have $g_M = Id + df^t \circ df$ and $\Gamma_f^* \omega_0 = -df + df^t$. The metric g_M is complete if f is defined on all \mathbb{R}^4 . The solutions of

$$\det(\Gamma_f^* \omega_0 - \lambda g_M) = 0 \quad (8.1)$$

are pure imaginary, and $\lambda^2 = -\cos^2 \theta_\alpha$ give the Kähler angles. We can compute explicitly (8.1). Set $f(x, y, z, w) = (u, v, s, t)$. Then

$$df = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} & \frac{\partial v}{\partial w} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} & \frac{\partial s}{\partial w} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} & \frac{\partial t}{\partial w} \end{bmatrix}$$

Now define

$$\begin{aligned} A &= -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & B &= \frac{\partial s}{\partial x} - \frac{\partial u}{\partial z} & C &= \frac{\partial t}{\partial x} - \frac{\partial u}{\partial w} & D &= \frac{\partial s}{\partial y} - \frac{\partial v}{\partial z} & E &= \frac{\partial t}{\partial y} - \frac{\partial v}{\partial w} & F &= \frac{\partial t}{\partial z} - \frac{\partial s}{\partial w} \\ l &= \langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial w} \rangle & m &= \langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w} \rangle & p &= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle & q &= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial z} \rangle & r &= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial w} \rangle & k &= \langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle \\ h &= (1 + \|\frac{\partial f}{\partial x}\|^2) & o &= (1 + \|\frac{\partial f}{\partial y}\|^2) & d &= (1 + \|\frac{\partial f}{\partial z}\|^2) & n &= (1 + \|\frac{\partial f}{\partial w}\|^2) \end{aligned}$$

$$\begin{aligned} \mathcal{A} &= 2hlkm + hodn - h(dl^2 + om^2 + nk^2) + p^2(-dn + m^2) + q^2(l^2 - no) + r^2(-od + k^2) \\ &\quad + 2qm(-lp + or) + 2pr(-mk + dl) + 2qk(pn - rl) \end{aligned}$$

$$\begin{aligned} \mathcal{B} &= 2DE(qr - hm) + 2BE(-rk + pm) + 2BD(lr - np) + 2CE(-dp + qr) \\ &\quad + 2AE(dr - qm) + 2CF(-oq + pk) + 2CB(-om + kl) + 2DF(-rp + hl) \\ &\quad + 2AF(-rk + ql) + 2AD(-rm + nq) + 2AC(-dl + mk) + 2AB(ml - nk) \\ &\quad + 2CD(-ql + mp) + 2FE(qp - kh) + 2FB(or - pl) + E^2(dh - q^2) \\ &\quad + B^2(no - l^2) + o(hF^2 + dC^2) + n(hD^2 + dA^2) - c^2k^2 - r^2D^2 - m^2A^2 - p^2F^2 \end{aligned}$$

$$\mathcal{D} = (AF - BE + CD)^2.$$

The Kähler angles of Γ_f are the solutions of (8.1) for $-\lambda^2 = \mu = \cos^2 \theta$ what explicitly reads $\mu^2 \mathcal{A} - \mu \mathcal{B} + \mathcal{D} = 0$. Thus Γ_f has e.k.a. iff $\mathcal{A} \neq 0$ and $\mathcal{B}^2 = 4\mathcal{A}\mathcal{D}$, and in this case $\cos^2 \theta = \frac{\mathcal{B}}{2\mathcal{A}} = \sqrt{\frac{\mathcal{D}}{\mathcal{A}}}$, or $\mathcal{A} = 0$ and in this case $\cos^2 \theta = \frac{\mathcal{D}}{\mathcal{B}}$. We can find a very large family of Cayley submanifolds M in a hyper-Kähler ambient space N by taking two different complex structures

J_x, J_y , and considering M J_x -complex and N J_y -complex, where x, y are any elements of S^2 . Those submanifolds are automatically minimal, and the expression of the k.a is simplified (see Prop.3.5). We will restrict ourselves to this case.

If we consider on \mathbb{R}^4 a g_0 -orthogonal complex structure J_ω , the complex structure $(J_\omega, -J_\omega)$ of \mathbb{R}^8 , $(J_\omega, -J_\omega)(X, Y) = (J_\omega X, -J_\omega Y)$ is g_0 -orthogonal and anti-commutes with J_0 . Then $(J_0, (J_\omega, -J_\omega), J_0 \times (J_\omega, -J_\omega))$ defines an Hyper-Kähler structure on \mathbb{R}^8 . If $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is a J_ω -anti-holomorphic map, then the graph of f is a $(J_\omega, -J_\omega)$ -complex submanifold of \mathbb{R}^8 . We consider J_ω the complex structure $i = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4)$, where e_1, e_2, e_3, e_4 is the canonic basis of $\mathbb{R}^4 \equiv \mathbb{R}^4 \times 0 \subset \mathbb{R}^8$. Recall that, considering \mathbb{C} with the usual complex structure, also denoted by J_0 , $J_0(x, y) = (-y, x)$, and if $f(x, y) = (u, v) : \mathbb{R}^2 \equiv \mathbb{C} \rightarrow \mathbb{R}^2 \equiv \mathbb{C}$ then $f(x, y) = (u, v)$ is anti-holomorphic iff $df \circ J_0 = -J_0 \circ df$, iff $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$, that is, iff $h(x, y) = (v, u)$ is holomorphic. Let $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $f(x, y, z, w) = (s, t, u, v)$. Then f is anti- i -holomorphic iff $(x, y) \rightarrow (u, v)$, $(x, y) \rightarrow (s, t)$, $(z, w) \rightarrow (u, v)$ and $(z, w) \rightarrow (s, t)$ are anti-holomorphic, iff

$$\begin{cases} \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} & \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} = -\frac{\partial v}{\partial w} & \frac{\partial u}{\partial w} = \frac{\partial v}{\partial z} \\ \frac{\partial s}{\partial x} = -\frac{\partial t}{\partial y} & \frac{\partial s}{\partial y} = \frac{\partial t}{\partial x} & \frac{\partial s}{\partial z} = -\frac{\partial t}{\partial w} & \frac{\partial s}{\partial w} = \frac{\partial t}{\partial z} \end{cases} \quad (8.2)$$

This implies

$$\begin{aligned} A = F = 0; \quad B = -E = \frac{\partial s}{\partial x} - \frac{\partial u}{\partial z}; \quad C = D = \frac{\partial s}{\partial y} - \frac{\partial u}{\partial w}; \quad p = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = 0; \quad m = \langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w} \rangle = 0; \\ q = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial z} \rangle = \frac{\partial u}{\partial x} \frac{\partial u}{\partial z} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial w} + \frac{\partial s}{\partial x} \frac{\partial s}{\partial z} + \frac{\partial s}{\partial y} \frac{\partial s}{\partial w} = l = \langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial w} \rangle \\ r = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial w} \rangle = -\frac{\partial u}{\partial y} \frac{\partial u}{\partial z} - \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial t}{\partial y} \frac{\partial t}{\partial z} - \frac{\partial s}{\partial y} \frac{\partial s}{\partial z} = -k = -\langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle \\ h = (1 + \|\frac{\partial f}{\partial x}\|^2) = (1 + \|\frac{\partial f}{\partial y}\|^2) = o; \quad d = (1 + \|\frac{\partial f}{\partial z}\|^2) = (1 + \|\frac{\partial f}{\partial w}\|^2) = n \end{aligned}$$

$$\begin{aligned} \mathcal{A} = (hd - q^2 - k^2)^2 \geq 1; \quad \mathcal{B} = 4BCkq + 2(B^2 + C^2)(dh - k^2 - q^2) \\ \mathcal{D} = (B^2 + C^2)^2; \quad \cos^2 \theta = \frac{B^2 + C^2}{hd - k^2 - q^2}. \end{aligned}$$

Note that, the linear map $(u_0, v_0) : \mathbb{R}^4 \rightarrow \mathbb{C} \equiv \mathbb{R}^2$

$$(u_0, v_0)(x, y, z, w) = (x + y + z + w, x - y + z - w) \quad (8.3)$$

is anti-holomorphic, considering $\mathbb{C} = \mathbb{R}^2$ and \mathbb{R}^4 with the complex structures J_0 and $i = J_0 \times J_0$, respectively, or equivalently, (u_0, v_0) satisfies the first eq. of (8.2)

Proposition 8.1. *If f is anti- i -holomorphic and at a point p , $r = k = 0$ that is, at p , $\frac{\partial u}{\partial y} \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x} \frac{\partial u}{\partial w} = -\frac{\partial s}{\partial y} \frac{\partial s}{\partial z} + \frac{\partial s}{\partial x} \frac{\partial s}{\partial w}$, then Γ_f is a minimal submanifold with e.k.a. θ at p given by*

$$\cos^2 \theta = \frac{(\frac{\partial s}{\partial x} - \frac{\partial u}{\partial z})^2 + (\frac{\partial s}{\partial y} - \frac{\partial u}{\partial w})^2}{(1 + \|\frac{\partial f}{\partial x}\|^2)(1 + \|\frac{\partial f}{\partial z}\|^2) - \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial z} \rangle^2}$$

Proof. We only have to apply the above formulas, and the fact that since f is anti- i -holomorphic $\frac{\partial f}{\partial x} = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial s}{\partial x}, \frac{\partial s}{\partial y})$, $\frac{\partial f}{\partial y} = (\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}, \frac{\partial s}{\partial y}, -\frac{\partial s}{\partial x})$, $\frac{\partial f}{\partial z} = (\frac{\partial u}{\partial z}, \frac{\partial u}{\partial w}, \frac{\partial s}{\partial z}, \frac{\partial s}{\partial w})$, $\frac{\partial f}{\partial w} = (\frac{\partial u}{\partial w}, -\frac{\partial u}{\partial z}, \frac{\partial s}{\partial w}, -\frac{\partial s}{\partial z})$ and so $\|\frac{\partial f}{\partial y}\| = \|\frac{\partial f}{\partial x}\|$. \square

Corollary 8.1. *If $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ satisfies*

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y} = \frac{\partial u}{\partial z} = -\frac{\partial v}{\partial w} & \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} = \frac{\partial v}{\partial z} = \frac{\partial u}{\partial w} \\ \frac{\partial s}{\partial x} &= -\frac{\partial t}{\partial y} = \frac{\partial s}{\partial z} = -\frac{\partial t}{\partial w} & \frac{\partial t}{\partial x} &= \frac{\partial s}{\partial y} = \frac{\partial t}{\partial z} = \frac{\partial s}{\partial w} \end{aligned} \quad (8.4)$$

Then f is in the conditions of Proposition 8.1 at every $p \in \mathbb{R}^4$, with $h = o = d = n$ and $q = l = \|\frac{\partial f}{\partial x}\|^2 = \|\frac{\partial f}{\partial z}\|^2 = (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial s}{\partial x})^2 + (\frac{\partial s}{\partial y})^2$, and

$$\cos^2 \theta = \frac{(\frac{\partial s}{\partial x} - \frac{\partial u}{\partial z})^2 + (\frac{\partial s}{\partial y} - \frac{\partial u}{\partial w})^2}{1 + 2((\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial s}{\partial x})^2 + (\frac{\partial s}{\partial y})^2)}$$

Γ_f is a complete Cayley submanifold with no J_0 -complex points. Furthermore, $(\frac{\partial s}{\partial x} - \frac{\partial u}{\partial z})^2 + (\frac{\partial s}{\partial y} - \frac{\partial u}{\partial w})^2$ is bounded iff $\cos^2 \theta$ is bounded by a constant $\delta < 1$.

Proof. Set $X = -\frac{\partial u}{\partial x}$, $Y = -\frac{\partial u}{\partial y}$, $a = (B - 2X)^2 + (C - 2Y)^2 = (\frac{\partial s}{\partial x} + \frac{\partial u}{\partial x})^2 + (\frac{\partial s}{\partial y} + \frac{\partial u}{\partial y})^2$, and $\zeta = B^2 + C^2$. Then $\cos^2 \theta = \frac{\zeta}{1+a+\zeta}$ with $\zeta, a \geq 0$. This function is an increasing function on ζ , what implies the last assertion. \square

Remark. For any constants α, β , the map $f = (\alpha(u_0, v_0), \beta(u_0, v_0))$, where (u_0, v_0) is given by (8.3), satisfies the conditions of Corol. 8.1.

The following example of [7] was announced in [20] (we note that in [20] is missing a squareroot on the denominator of the expression of $\cos \theta$). It is an example on the conditions of Cor.8.1.

Proposition 8.2. *([D-S], [S,1]) Let $\phi(t) = \sin(t)$, $\xi(t) = \sinh(t)$, and*

$$\begin{aligned} u(x, y, z, w) &= \phi(x+z)\xi'(y+w) \\ v(x, y, z, w) &= -\phi'(x+z)\xi(y+w) \end{aligned} \quad (8.5)$$

then:

- (a) *If $f = (u, v, u, v)$, Γ_f is a complete minimal Lagrangian submanifold.*
- (b) *If $f = (u, v, -u, -v)$, Γ_f is a complete Cayley submanifold with e.k.a and*

$$\cos \theta = 2\sqrt{\frac{\cos^2(x+z) + \sinh^2(y+w)}{1 + 4(\cos^2(x+z) + \sinh^2(y+w))}} \quad (8.6)$$

Thus Γ_f has no J_0 -complex points, but $\cos \theta$ assume all values of $[0, 1[$. The set of Lagrangian points is an infinite discrete family of parallel 2-planes $\mathcal{L} = \bigcup_{k \in \mathbb{Z}} \mathbb{R} \cdot (1, 0, -1, 0) \oplus \mathbb{R} \cdot (0, 1, 0, -1) + (0, 0, \frac{\pi}{2} + k\pi, 0)$.

Proposition 8.3. *Let $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be a map.*

- (1) *A point p_0 is a J_0 -complex point of Γ_f iff $df(p_0) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is a complex structure of \mathbb{R}^4 . If that is the case, then it is g_M -orthogonal. It is g_0 -orthogonal iff $g_M = 2g_0$.*
- (2) *If J_ω is a g_0 -orthogonal complex structure of \mathbb{R} and at a point p_0 , $df(X) = aJ_\omega(X)$ where a is any nonzero real number, then Γ_f has e.k.a. at a point p_0 , with $\cos \theta = \frac{2|a|}{1+a^2}$ and $\Gamma_f^* \omega_0(X, Y) = g_M(\cos \theta \epsilon J_\omega(X), Y)$, where J_ω is also a g_M -orthogonal structure on \mathbb{R}^4 , and $\epsilon = \text{sign } a$.*

Proof. (1) At p_0 , Γ_f is a J_0 -complex submanifold iff $\forall X \in \mathbb{R}^4 \exists Y \in \mathbb{R}^4$ s.t. $(Y, df(Y)) = J_0(X, df(X)) = (-df(X), X)$. that is $-df(df(X)) = X$, but this is equivalent to $df(p_0) : \mathbb{R}^4 \rightarrow$

\mathbb{R}^4 to be a complex structure J_ω . Now $g_M(X, Y) = g_0(X, Y) + g_0(J_\omega(X), J_\omega(Y))$ and so g_M is J_ω -Hermitian, or equivalently J_ω is g_M -orthogonal. Now easily follows that J_ω is g_0 -orthogonal iff $g_M = 2g_0$.

(2) the condition e.k.a, $\Gamma_f \omega_0 = \cos \theta J_\omega$ (under a g_M -musical isomorphism), means $-g_0(df(X), Y) + g_0(X, df(Y)) = \cos \theta g_0(J_\omega(X), Y) + \cos \theta g_0(df(J_\omega X), df(Y))$ for some J_M -orthogonal structure J_ω on \mathbb{R}^4 . Obviously if $df(p_0) = aJ_\omega$ with J_ω g_0 -orthogonal, then immediately we verify that Γ_f has e.k.a. at p_0 with $\cos \theta = \frac{2|a|}{1+a^2}$ and $\Gamma_f^* \omega_0 = \cos \theta \epsilon J_\omega$. \square

Corollary 8.2. *Assume f is anti- i -holomorphic and Γ_f is at a point p_0 a J_0 -complex submanifold of \mathbb{R}^8 , that is $df(p_0) = J_\omega$ where J_ω is a complex structure of \mathbb{R}^4 , g_M -orthogonal. Then i and J_ω anti-commute and are both g_M -orthogonal on \mathbb{R}^4 .*

Proof. Let $X, Y \in \mathbb{R}^4$. From $df(p_0)(iX) = -idf(p_0)(X)$ we have $J_\omega \circ i = -i \circ J_\omega$. That is, J_ω and i anti-commute. Now, at p_0 $g_M(iX, iY) = g_0(iX, iY) + g_0(df(p_0)(iX), df(p_0)(iY)) = g_0(X, Y) + g_0(-idf(p_0)(X), -idf(p_0)(Y)) = g_M(X, Y)$. So i is also g_M -orthogonal. \square

Now we are ready to get examples of non- J_0 -holomorphic Cayley submanifolds of (\mathbb{R}^8, J_0, g_0) with J_0 -complex points, or non-linear Cayley graphs with $\cos \theta \leq \delta < 1$.

Consider the complex structure j of \mathbb{R}^4 $j = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_2 \wedge e_4)$. Then $ji = -ij$, so j can be seen as a (linear) anti- i -holomorphic map of \mathbb{R}^4 , $j(x, y, z, w) = (-z, w, x, -y)$ (with $dj(p) = j \nabla p = (x, y, z, w)$).

Proposition 8.4. *Let $\tilde{f} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be any anti- i -holomorphic with $d\tilde{f}(0) = 0$. Then $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $f = j + \tilde{f}$ is s.t. its graph defines a Cayley submanifold of \mathbb{R}^8 with a J_0 -complex point 0.*

Proof. Since $df(0) = j$, by proposition 8.3(1) the tangent space of Γ_f is at 0 a J_0 -complex subspace of \mathbb{R}^8 . \square

Corollary 8.3. (1) $f(x, y, z, w) = j(x, y, z, w) + (x^2 - y^2, -2xy, z^2 - w^2, -2zw)$ defines a non J_0 -holomorphic Cayley submanifold of \mathbb{R}^8 with only one J_0 -complex point, namely at 0.

(2) $f(x, y, z, w) = j(x, y, z, w) + (x^2 - y^2, -2xy, 0, 0)$ defines a non J_0 -holomorphic Cayley submanifold of \mathbb{R}^8 with set of J_0 -complex point $\mathcal{C} = \mathbb{R}^2 \times \{(0, 0)\}$.

Proof. (1) From previous proposition 0 is a J_0 -complex point. If $p = (x, y, z, w)$ is a J_0 -complex point of Γ_f , then $df(p) = j + \xi$, with $(j + \xi)^2 = -Id$, where

$$\xi = (2xe_*^1 - 2ye_*^2, -2ye_*^1 - 2xe_*^2, 2ze_*^3 - 2we_*^4, -2we_*^3 - 2ze_*^4)$$

From $-Id = (j + \xi)^2 = j^2 + j\xi + \xi j + \xi^2 = -Id + j\xi + \xi j + \xi^2$, we should have $j\xi + \xi j = -\xi^2$. But $\xi^2 = 4(x^2 + y^2)(e_*^1 \otimes e_1 + e_*^2 \otimes e_2) + 4(z^2 + w^2)(e_*^3 \otimes e_3 + e_*^4 \otimes e_4)$ and $(j\xi + \xi j)(e_1) = 2(x + z)e_3 + 2(y - w)e_4$, $(j\xi + \xi j)(e_3) = -2(x + z)e_1 + 2(y - w)e_2$, and so $j\xi + \xi j = -\xi^2$ is only possible for $p = 0$. The case (2) is similar with $\xi = (2xe_*^1 - 2ye_*^2, -2ye_*^1 - 2xe_*^2, 0, 0)$. \square

Proposition 8.5. *Let $\tilde{f} = f + (\alpha(u_0, v_0), \beta(u_0, v_0))$ where $f = (u, v, u, v)$ is given by Prop.8.2(a) and (u_0, v_0) by (8.3) and α, β any constants. Then \tilde{f} is anti- i -holomorphic satisfying (8.4), and $\cos \theta \leq \frac{2(\alpha^2 + \beta^2)}{1 + 2(\alpha^2 + \beta^2)} < 1$.*

Proof. Use proof of Cor. 8.1 to check the upper bound of $\cos \theta$. \square

8.2 Coassociative graphs

A coassociative graph is a Cayley graph of a map $f : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \subset \mathbb{R}^4$ (see [12]). In this case at each point p , $df(p) : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \subset \mathbb{R}^4$ cannot be an isomorphism. Thus, by Prop. 8.3 (1) we have:

Corollary 8.4. *If Γ_f is a coassociative graph then it has no J_0 -complex points.*

An example of a coassociative graph given in [12] is the graph of $\eta : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ $\eta(x) = \frac{\sqrt{5}}{2\|x\|} \bar{x} \epsilon x$ where the product is the quaternionic product and ϵ is a unit of $\mathbb{R}^3 = \text{Im} \mathbb{R}^4$. This is the cone of the Hopf map from S^3 to S^2 .

References

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